Single-Node MMSE for MMSE Cooperative Positioning

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ABSTRACT

We proposed in our previous work an iterative minimum-mean-square-error (MMSE) cooperative positioning algorithm. MMSE cooperative positioning method achieves better root-mean-square-error (RMSE) performance than existing classical estimators. And it is implemented in an iterative pattern so as to circumvent the intense computation burden of the numerical multiple integral computation methods. The basis of the proposed iterative MMSE method is the single-node MMSE, which is actually the special case of the MMSE cooperative method when the number of node N being 1. In this work, we study the properties of the single-node MMSE and accordingly propose three variants of the original algorithm to improve the performance. The single-node MMSE and its variants can also be used to produce initial position estimation for the maximum likelihood estimator (MLE), one of the most popular existing classical estimators, and achieve almost the same performance as using true positions as the initial positions.

Keywords: cooperative positioning, MMSE

1. INTRODUCTION

We proposed in our previous work[1]. minimum-mean-square-error (MMSE) cooperative positioning algorithm. Numerical methods are needed for computing the multiple integrals in MMSE formula. This brings intense computation burden especially for large size networks. To circumvent this problem, we also proposed in[1] to implement MMSE cooperative positioning algorithm in an iterative pattern where only a single-node MMSE needs to be computed at each iteration. The iterative MMSE costs a lot less computation and achieves better root-mean-square-error (RMSE) performance than existing classical estimators.

Obviously, the single-node MMSE, a special case of the MMSE cooperative method when the number of node N being 1, is the basis of the iterative MMSE. In this work, we study the properties of the single-node MMSE and propose three variants to improve the performance.

One of the significant properties of MMSE estimator is that better performance is achieved for nodes that are closer to a priori probability density function (PDF) center. Another important property observed is that for a square area, although the true positions are scattered all over the whole square area, the positions estimated by the original MMSE estimator fall within a squeezed-box shape area. Greater details are presented later.

According to these properties, three variants are proposed. The first variant is named large scale MMSE (LS-MMSE), where a virtual a priori PDF that covers a larger area than the actual a priori PDF is used. The second variant is referred to as MMSE-mapping, where we map the originally obtained estimated position to another position according to certain rules and use the position after mapping as the final estimated positions. The third variant, called two-stage MMSE (TS-MMSE) is to add one more step to MMSE-mapping. At this second step, original MMSE is applied again assuming that each true position is uniformly distributed within a new smaller square whose center is the obtained estimated position.

Although these variants are proposed for single-node MMSE, it is actually quite straightforward to extend the application of these modified MMSE estimator to multiple nodes networks.

The single-node MMSE and its variants can also be used to produce initial position estimation for the maximum likelihood estimator (MLE), one of the most popular existing classical estimators, and achieve almost the same performance as using true positions as the initial positions.

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The remaining of this paper is organized as follows. The system model and MMSE cooperative positioning algorithm are presented in Section 2. Then, we take a close look at the properties of the performance of the single-node MMSE algorithm in Section 3. Based on these properties, we make modifications accordingly to the original single-node MMSE algorithm and propose three variants, as presented in Section 4. Numerical results shown in Section 5 verify the improved performance achieved by these variants. Conclusion of this paper is then addressed in Section 6.

2. SYSTEM MODEL AND MMSE COOPERATIVE POSITIONING

The nodes whose positions are unknown and to be estimated are referred to as *unknown nodes.* To estimate the coordinates of unknown nodes, we need *anchor nodes*, whose positions are known in advance.

Consider a wireless network of N unknown nodes and M anchor nodes. The position for any unknown node i, $1 \le i \le N$, is described by its coordinates (x_i, y_i). The node $j = N+1, \cdots, M$ refers to one of M anchor nodes. All nodes transmit at a fixed known power level. The received signal power between any pair of nodes is observed to estimate the coordinates of unknown nodes.

Let P_{ij} denote the received power strength at the node i from the node j, whose distance is denoted as d_{ij}. As in,[2] we adopt the classical log-normal distribution[3] for P_{ij}. Thus, we have

$$P_{ij}(dB) \sim \mathcal{N}\left(\bar{P}_{ij}(dB), \sigma_{dB}^2\right) \tag{1}$$

where $\bar{P}_{ij}(dB)$ is the expectation corresponding to the specific d_{ij} and the variance σ_{dB}^2 keeps the same for any distance. Suppose the average received power strength at distance d_0 is $P_0(dB)$. P_0 and d_0 are called the reference power and distance respectively. According to,[3] we have

$$\bar{P}_{ij}(dB) = P_0(dB) - 10 n_p log_{10}\left(\frac{d_{ij}}{d_0}\right), \tag{2}$$

where n_p is the *path loss exponent.*

As in,[4] the a priori distribution for unknown nodes is assumed to be independent uniform distribution. Suppose the node i may appear within a rectangular box centered at (O_{ix}, O_{iy}) of $2A_i$ long along x-axis and $2B_i$ long along y-axis. Then, for $x_i \in (O_{ix} - A_i, O_{ix} + A_i), y_i \in (O_{iy} - B_i, O_{iy} + B_i)$, $1 \le i \le N$, the a priori PDF is expressed as

$$f(x_1, y_1, \cdots, x_N, y_N) = \prod_{i=1}^{N} \frac{1}{2A_i} \prod_{i=1}^{N} \frac{1}{2B_i}. \tag{3}$$

As presented in our previous work,[1] with the independent uniform a priori distribution (3) and the log-normal distribution for power degradation (1), the MMSE cooperative position estimator for node i, $1 \le i \le N$, can be derived as

$$\begin{aligned}
\hat{x}_{i,\text{MMSE}} &= \frac{\int_{S_N} \cdots \int_{S_1} x_i \prod_{i=1}^{N} \prod_{j=i+1}^{N+M} \exp\left(-\frac{\alpha}{8} \ln^2 \frac{d_{ij}^2}{\tilde{d}_{ij}^2}\right) d\boldsymbol{\theta}_1 \cdots d\boldsymbol{\theta}_N}{\int_{S_N} \cdots \int_{S_1} \prod_{i=1}^{N} \prod_{j=i+1}^{N+M} \exp\left(-\frac{\alpha}{8} \ln^2 \frac{d_{ij}^2}{\tilde{d}_{ij}^2}\right) d\boldsymbol{\theta}_1 \cdots d\boldsymbol{\theta}_N} \\
\hat{y}_{i,\text{MMSE}} &= \frac{\int_{S_N} \cdots \int_{S_1} y_i \prod_{i=1}^{N} \prod_{j=i+1}^{N+M} \exp\left(-\frac{\alpha}{8} \ln^2 \frac{d_{ij}^2}{\tilde{d}_{ij}^2}\right) d\boldsymbol{\theta}_1 \cdots d\boldsymbol{\theta}_N}{\int_{S_N} \cdots \int_{S_1} \prod_{i=1}^{N} \prod_{j=i+1}^{N+M} \exp\left(-\frac{\alpha}{8} \ln^2 \frac{d_{ij}^2}{\tilde{d}_{ij}^2}\right) d\boldsymbol{\theta}_1 \cdots d\boldsymbol{\theta}_N}
\end{aligned} \tag{4}$$

where ln is the natural logarithm and

$$\int \cdot \, d\boldsymbol{\theta}_i \equiv \int\int \cdot \, dx_i dy_i. \tag{5}$$

S_i is the integral region for $\boldsymbol{\theta}_i = (x_i, y_i)$, expressed as

$$S_i = \left\{ (x_i, y_i) \middle| \begin{array}{l} x_i \in (O_{ix} - A_i, O_{ix} + A_i) \\ y_i \in (O_{iy} - B_i, O_{iy} + B_i) \end{array} \right\}. \tag{6}$$

$$\alpha = \left(\frac{10 n_p}{\sigma_{dB} \ln 10} \right)^2, \tag{7}$$

and

$$\hat{d}_{ij} = d_0 \left(\frac{P_0}{P_{ij}} \right)^{1/n_p}. \tag{8}$$

Numerical methods, such as Simpon quadrature and Monte Carlo methods[5] , are needed to compute multiple integrals in MMSE formula (4), which are complicated and have no close form solution. The computation cost increases exponentially with the number of unknown nodes. To circumvent huge computation burden, we propose an iterative MMSE in[1] where the MMSE formula (4) for the special case $N = 1$ is repeatedly applied with adaptively adjusted priori PDF.

It is obvious that the overall performance depends on the performance of single-node MMSE. Therefore, in the following, we take a close look at the performance and properties of single-node MMSE and explore modifications to achieve better performance.

3. PERFORMANCE AND PROPERTIES OF SINGLE-NODE MMSE

As in,[6,7] it is assumed that all unknown nodes are within a single square area and there are four anchor nodes, with one anchor node at each of the four corners. Now, let us have a look at how the MMSE algorithm performs in the case of single-node i.e. $N = 1$.

The root mean square error (RMSE) between the true and estimated positions is used as the performance metric. We are going to look at two kinds of RMSE. Average RMSE, which is averaged with respect to all possible true positions, is used to evaluate overall performance of certain positioning algorithm. The RMSE for a particular true position $(\tilde{x}, \tilde{y})$, denoted as RMSE$(\tilde{x}, \tilde{y})$, provides insights into how different true positions contribute differently to the average RMSE. RMSE$(\tilde{x}, \tilde{y})$ can be mathematically expressed as

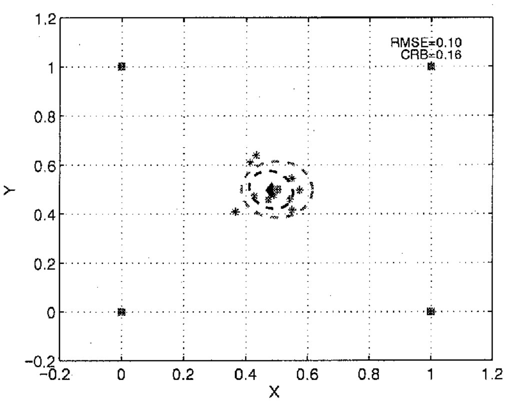
$$\text{RMSE}(\tilde{x}, \tilde{y}) = \sqrt{\mathsf{E}[(\tilde{x} - \hat{\tilde{x}})^2 + (\tilde{y} - \hat{\tilde{y}})^2 | (\tilde{x}, \tilde{y})]}, \tag{9}$$

where the expectation is taken only with respect to $(\hat{\tilde{x}}, \hat{\tilde{y}})$.

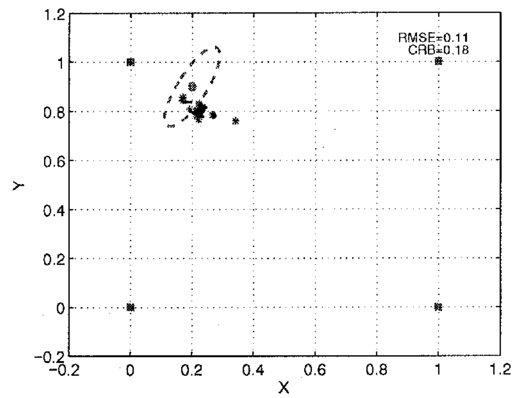
The performance of the single-node MMSE for different true positions is illustrated in Fig. 1. We study RMSE$(\tilde{x}, \tilde{y})$ for three different types of positions:(a) *center position*, the position close to a priori PDF center, which is $(0.5, 0.5)$ in our example; (b) *corner position*, a position near a corner; (c) *side position*, a position besides a side. As an example, in Fig. 1, the corner and side positions are picked at $(0.2, 0.9)$ and $(0.1, 0.5)$ respectively.

As shown in Fig. 1,the RMSE for the center positions is smaller than that for those positions further away from a priori PDF center. This is expected since MMSE generally works better when true value gets closer to the expectation.[8] Furthermore, among the positions far away from a priori PDF center, the RMSE for corner positions is smaller than that for side positions. Intuitive explanations are as follows. The MMSE estimator for a corner position is inherently restricted into a smaller area than that for a side position and is thus subject to smaller possible errors. The Cramér-Rao bound (CRB) is also shown in the figure for comparison purpose.

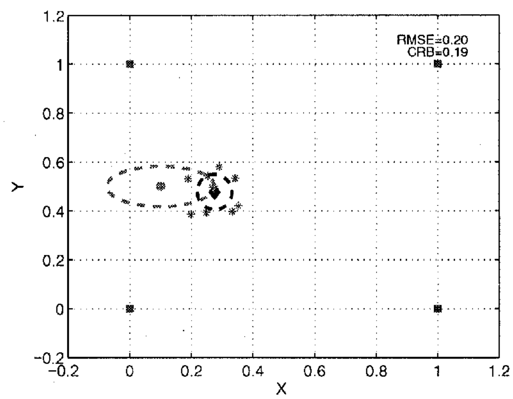
To have a complete picture about how the MMSE estimator performs for single-node, 500 positions are randomly picked as the true positions for $N = 1$ case and the result is shown in Fig. 2. It is noticed that though the true positions are uniformly distributed among the whole square, the estimated positions tend to fall within a squeezed-box shape area as covered by the blue or the black diamonds. This means that bias of MMSE estimator is much larger for those side positions. These observations motivate the variants of the MMSE estimator, which will be presented in the next section.



(a) Center position: RMSE$(0.5, 0.5) = 0.10$



(b) Corner position: RMSE$(0.2, 0.9) = 0.11$



(c) Side position: RMSE$(0.1, 0.5) = 0.2$

Figure 1. The single-node MMSE performance for different true positions (See Table 1 for symbol explanations.)

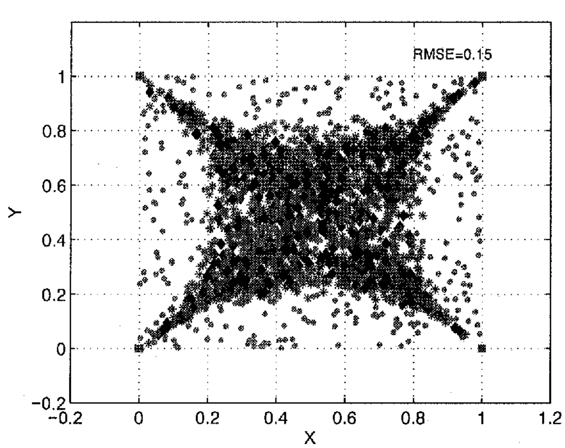


Figure 2. The single-node MMSE performance (See Table 1 for symbol explanations.)

Table 1. Symbol Explanation for Fig. 1~2

Symbol	Meaning
blue square	anchor nodes
red dot	the true position
blue asterisk	estimated position
black diamond	the mean of the estimated positions for a true position
black ellipse	uncertainty ellipse
red ellipse	CRB ellipse

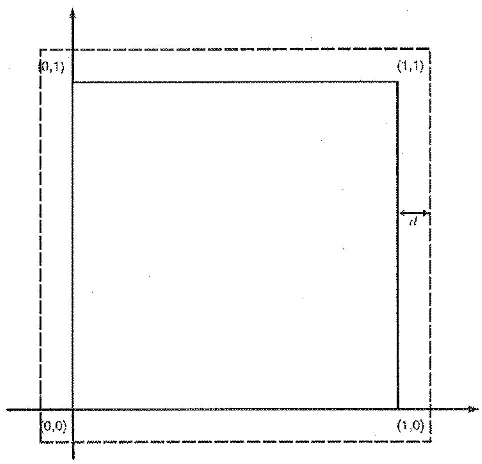


Figure 3. LS-MMSE estimator uses the larger square (dashed line) for the virtual a priori PDF, while the smaller square (solid line) is for the actual a priori PDF.

4. VARIANTS OF SINGLE-NODE MMSE ALGORITHM

4.1 LS-MMSE

According to the section 3, better RMSE is achieved by the original MMSE estimator for nodes that are closer to a priori PDF center. This enlightened the thought that if we could "push" all nodes closer to the center then the overall performance may be improved. Since a priori PDF for the true positions, or the *actual a priori PDF*, is fixed, we cannot actually push the nodes to be closer to the center. However, we can consider a larger square for a priori PDF used for computing the conditional mean. This equivalently brings all nodes relatively closer to the center. This a priori PDF with larger square is called *virtual a priori PDF*. Specifically, as sketched in Fig. 3, though the true position (x, y) is distributed according to uniform distribution within a 1 by 1 square (solid line), the uniform distribution within a larger $1+2d$ by $1+2d$ square (dashed line) is used instead for computing integrals for the conditional mean. The resulting MMSE estimator is named *large scale MMSE* (LS-MMSE) estimator, which is obtained according to (4) with

$$S_i = \{(x_i, y_i)| -d \leq x_i \leq 1+d, -d \leq y_i \leq 1+d\}, \forall i.$$

An empirical value for d is one tenth of the side length. In our case, $d = 0.1$. LS-MMSE improves the RMSE performance (as shown by the simulation results later) without any extra computation burden. It is noted that in the original MMSE estimator, the actual a priori PDF is the same as the virtual a priori PDF.

4.2 MMSE-Mapping and TS-MMSE

Fig. 2 in the section 3 reveals that though the true positions are scattered all over the whole square area, the positions estimated by the original MMSE estimator fall within a squeezed-box shape area. Based on this

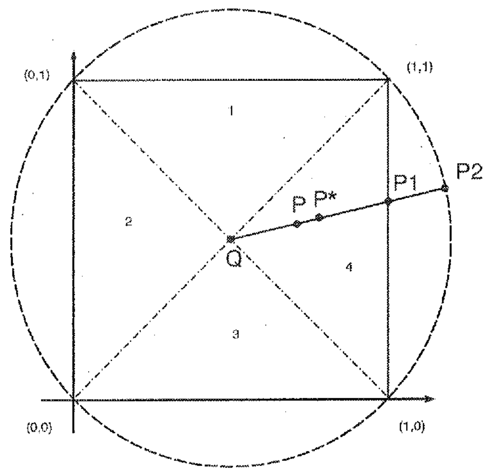


Figure 4. The mapping scheme for MMSE-Mapping

observation, we propose to map the MMSE estimated position to a new position so that the area covered by the estimated positions after mapping can overlap with the area covered by the true positions as much as possible. Intuitively, this will bring the average estimated positions closer to the true position and thus can reduce bias and RMSE. The obtained estimator is named MMSE-Mapping.

The mapping is illustrated in Fig. 4. The square is the original area where all nodes appear and it is divided into four regions marked from 1 to 4. The circumcircle is introduced for mapping. P at $(\hat{x}, \hat{y})$ is the original MMSE estimated position and P^* at $(\hat{x}^*, \hat{y}^*)$ is the new estimated position after mapping. An auxiliary line, connecting the center point Q at $(1/2, 1/2)$ and P, intersects the square at P_1 and the circumcircle at P_2. We choose P^* so that

$$\frac{d(P^*, Q)}{d(P, Q)} = \frac{d(P_2, Q)}{d(P_1, Q)}, \tag{10}$$

where $d(A, B)$ stands for the distance between the point A and the point B. Obviously, $d(P_2, Q) = \sqrt{2}/2$, the radius of the circumcircle. Let k be the slope of the auxiliary line, thus $k = \frac{\hat{y}-\frac{1}{2}}{\hat{x}-\frac{1}{2}}$. The coordinates of P_1, (x_{P_1}, y_{P_1}), which are needed to compute $d(P_1, Q)$, are

$$(x_{P_1}, y_{P_1}) = \begin{cases} (\frac{1}{2} + \frac{1}{2k}, 1) & \text{if } P_1 \text{ is in the region 1} \\ (0, \frac{1}{2} - \frac{k}{2}) & \text{if } P_1 \text{ is in the region 2} \\ (1, \frac{1}{2} + \frac{k}{2}) & \text{if } P_1 \text{ is in the region 3} \\ (\frac{1}{2} - \frac{1}{2k}, 0) & \text{if } P_1 \text{ is in the region 4} \end{cases}. \tag{11}$$

According to the mapping rule in (10), it is easy to derive that

$$\begin{cases} \hat{x}^* = \frac{1}{2} + \frac{\sqrt{2}}{2} \frac{d(P,Q)}{d(P_1,Q)} \cos\theta \\ \hat{y}^* = \frac{1}{2} + \frac{\sqrt{2}}{2} \frac{d(P,Q)}{d(P_1,Q)} \sin\theta \end{cases} \tag{12}$$

where, $d(P, Q) = \sqrt{(\hat{x} - 1/2)^2 + (\hat{y} - 1/2)^2}$, $d(P_1, Q) = \sqrt{(x_{P_1} - 1/2)^2 + (y_{P_1} - 1/2)^2}$ with (x_{P_1}, y_{P_1}) obtained according to (11), and $\theta = \arg\tan \frac{\hat{y}-\frac{1}{2}}{\hat{x}-\frac{1}{2}}$, which is the angle of the auxiliary line.

Performance can be further improved if we apply the MMSE estimation method again, assuming that each true position is uniformly distributed within a new smaller square whose center is the corresponding MMSE-Mapping estimator, i.e. the estimated position after mapping. For this second stage MMSE, empirically, the length of the smaller square can be chose as one fifth of the length of the original square. The obtained new estimator is named two-stage MMSE (TS-MMSE) estimator, since we apply MMSE twice at two stages. To put

Table 2. Comparison of Different Estimations

Estimator	RMSE
MDS	0.2571
MLE	0.2031
MDS-MLE	0.1959
MLE-Ideal	0.1928
CRB=0.1811	
MMSE	0.1515
MMSE-Mapping	0.142
LS-MMSE	0.1344
TS-MMSE	0.1329

MLE : MLE with random initial estimation
MLE-Ideal : MLE with true positions as initial estimation

it precisely, the TS-MMSE estimator is obtained using (4), where $A_i = B_i = 0.2$ and $(O_{ix}, O_{iy}) = (\hat{x}_i^*, \hat{y}_i^*)$ with $(\hat{x}_i^*, \hat{y}_i^*)$ being obtained according to (12) (the mapping step) from the original MMSE estimator $(\hat{x}_i, \hat{y}_i)$. The TS-MMSE works like a turbo engine. After the first time MMSE is implemented, mapping is carried out to tune up the estimated positions. These new estimated positions are used to determine the centers of smaller square areas so that MMSE can be implemented again with the new virtual a priori PDF.

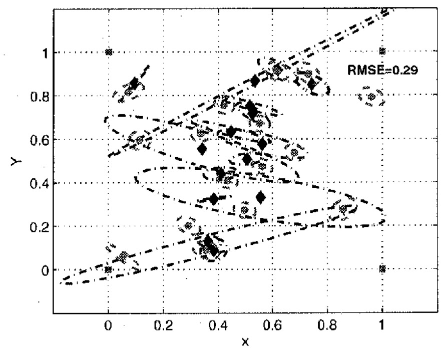
As pointed out in our introduction, the proposed variants can be easily applied to multiple node networks, though they are based on single-node MMSE. For example, if we replace all integral intervals corresponding to the actual a priori PDF in the cooperative MMSE formula (4) with intervals corresponding to the larger area of the virtual a priori PDF, we have LS-MMSE for multiple nodes positioning. Similarly, for MMSE-Mapping, when there are more than one nodes, we can apply mapping to each node's estimated position which is obtained using the cooperative MMSE algorithm. Then, we can use the estimated position as new center and apply cooperative MMSE algorithm again to obtain the TS-MMSE results.

5. NUMERICAL RESULTS

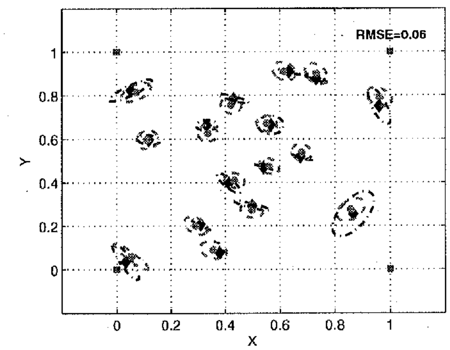
First, we compare original single-node MMSE and its variants with existing popular algorithms MLE, MDS in terms of RMSE and list the results in Tab. 2 where "MLE" refers to the regular MLE with random initial estimation and "MLE-Ideal" refers to the MLE algorithm with the true positions as the perfect initial estimation, which is the ideal case for MLE. The CRB is also provided for comparison. It is seen that RMSE of MMSE is smaller than CRB. This indicates MMSE is better than any (unbiased) classical estimator. Also, three variants achieves smaller RMSE than the original MMSE algorithm.

The well known MLE positioning estimator is quite sensitive to the initial estimation. It does not work well enough if the initial estimation is randomly generated. This can be seen from Tab. 2 and it is also shown in Fig. 5(a). If the perfect initial estimation, i.e. the true position, is used, MLE works extremely well as shown in Fig. 5(b). True positions are unknown and to be estimated, so MLE with perfect initial estimation is an ideal but impractical solution. Fortunately, a very good initial estimation can be obtained quickly by applying the single-node MMSE to each of the nodes. The obtained MMSE initial estimation is then fed into the iterative algorithm for MLE. The resulting estimator is then named MMSE-MLE. To improve the accuracy of initial estimator, we can implement certain variant of MMSE, such as LS-MMSE (Section 4.1), instead of the original MMSE. The performance of MMSE-MLE using LS-MMSE is shown in Fig. 5(c). It can be seen that MMSE-MLE performs as well as MLE with perfect initial estimation.

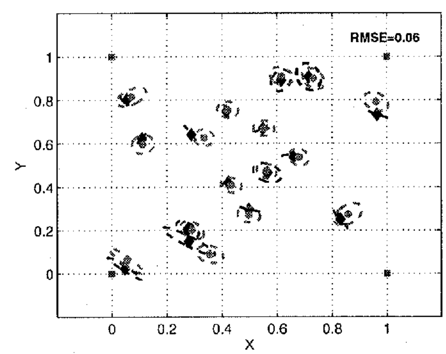
Now let's look at how the variants work for a more general case where $N > 1$. Of all three variants, LS-MMSE achieves the best tradeoff between performance improvement and extra computation cost. Therefore we focus on LS-MMSE. RMSE performance of different estimators for multiple nodes networks is shown in Fig.6, where the proposed MMSE cooperative estimator (MMSE), its variant LS-MMSE, MDS, MLE with random initial estimation (MLE), MLE with MDS used as the initial estimation (MDS-MLE) are compared. CRB is also



(a) MLE with random initial estimation



(b) MLE with perfect initial estimation



(c) MLE with LS-MMSE initial estimation (MMSE-MLE)

Figure 5. MMSE-MLE performs as well as MLE with perfect initial estimation ($N = 20$). Same symbols as given in Table 1 are used.

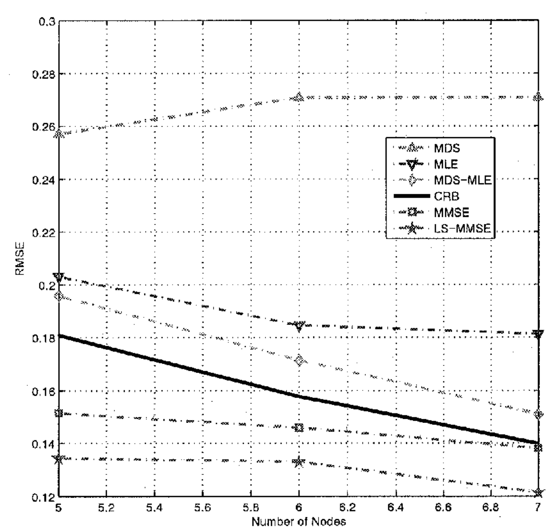


Figure 6. RMSE performance of different estimators for multiple nodes networks

provided. It can be seen that LS-MMSE brings obvious performance improvement over MMSE for any N and they both are better than the CRB.

6. CONCLUSIONS

We study how the proposed MMSE cooperative estimator works for the special case of $N = 1$. Based on the observed properties, three variants of the original MMSE estimator, which are LS-MMSE, MMSE-Mapping and TS-MMSE are proposed. And it is shown that they achieve better performance than original MMSE. Comparison with the most popular existing algorithms, including MDS and MLE, in terms of RMSE and CRB verifies the superior performance of the proposed MMSE estimator and its variants. The proposed MMSE estimator and its variants can also be used to provide initial position estimation for MLE positioning algorithm and achieve performance almost the same as using perfect initial estimation.

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