ELC 5396: Digital Communications

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Central Limit Theorem

Random variables $x_1(\zeta), x_2(\zeta), \ldots, x_M(\zeta)$

- (a) are mutually independent and
- (b) have the same distribution, and
- (c) the mean and variance of each random variable exist and are finite

Then, the distribution of the normalized sum

$$y_M(\zeta) = \frac{\sum_{k=1}^{M} x_k(\zeta) - \mu_{y_M}}{\sigma_{y_M}}$$

approaches that of a normal random variable with zero mean and unit standard deviation as $M \to \infty$.

Discrete-time Stochastic Processes

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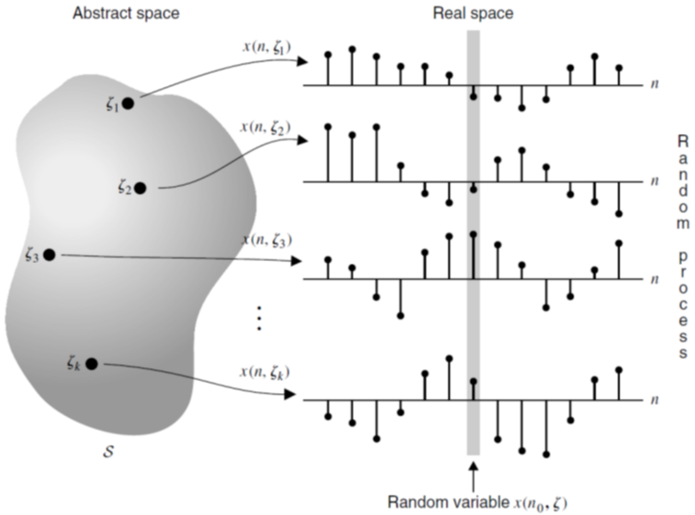
- The set of all possible sequences $\{x(n, \zeta)\}$ is called an ensemble, and each individual sequence $x(n, \zeta_k)$, corresponding to a specific value of $\zeta = \zeta_k$, is called a realization or a sample sequence of the ensemble.

Random Process



Random Process

- $x(n, \zeta)$ is a random variable if n is fixed and ζ is a variable.
- $x(n, \zeta)$ is a sample sequence if ζ is fixed and n is a variable.
- $x(n, \zeta)$ is a number if both n and ζ are fixed.
- $x(n, \zeta)$ is a stochastic process if both n and ζ are variables.

Features of Random Signals

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3. The existence of "cycles" or quasi-periodic patterns, obtained from the signal power spectrum.
4. Indications of variability in the mean, variance, probability density, or spectral content.

Description Using Probability Functions

The kth-order cdf:

$$F_x(x_1, \ldots, x_k; n_1, \ldots, n_k) = Pr\{x(n_1) \leq x_1, \ldots, x(n_k) \leq x_k\}$$

or the kth-order pdf:

$$f_x(x_1, \ldots, x_k; n_1, \ldots, n_k) = \frac{\partial^{2k} F_x(x_1, \ldots, x_k; n_1, \ldots, n_k)}{\partial x_{R1} \partial x_{I1} \cdots \partial x_{Rk} \partial x_{Ik}}$$

needs to be known for every value of $k \geq 1$ and for all instances $n_1, n_2, \ldots, n_k$.

Second-Order Statistical Description

The second-order statistic of $x(n)$ at time n is specified by its mean value $\mu_x(n)$ and its variance $\sigma_x^2(n)$, defined by

- $\mu_x(n) = E\{x(n)\} = E\{x_R(n) + jx_I(n)\}$
- $\sigma_x^2(n) = E\{|x(n) - \mu_x(n)|^2\} = E\{|x(n)|^2\} - |\mu_x(n)|^2$

respectively.

In general, both $\mu_x(n)$ and $\sigma_x(n)$ are deterministic sequences.

Autocorrelation and Autocovariance

The second-order statistics of $x(n)$ at two different times n_1 and n_2 are given by the two-dimensional autocorrelation (or autocovariance) sequences.

- $r_{xx}(n_1, n_2) = E\{x(n_1)x^*(n_2)\}$
- $\gamma_{xx}(n_1, n_2) = r_{xx}(n_1, n_2) - \mu_x(n_1)\mu_x^*(n_2)$

Cross-correlation and Cross-covariance

- $r_{xy}(n_1, n_2) = E\{x(n_1)y^*(n_2)\}$
- $\gamma_{xy}(n_1, n_2) = r_{xy}(n_1, n_2) - \mu_x(n_1)\mu_y^*(n_2)$

IID Random Sequence and Uncorrelated Processes

- A sequence of independent random variables. If all random variables have the same pdf $f(x)$ for all n_k, then $x(n)$ is called an IID (independent and identically distributed) random sequence.

- An uncorrelated process

$$\gamma_x(n_1, n_2) = 0 \quad \text{if} \quad n_1 \neq n_2$$

Orthogonal Process and WSS Process

- An orthogonal process

$$r_x(n_1, n_2) = 0 \text{ if } \; n_1 \neq n_2$$

- A wise-sense cyclostationary process

$$\mu_x(n) = \mu_x(n + N), \;\; \forall n$$

$$r_x(n_1, n_2) = r_x(n_1 + N, n_2 + N), \;\; \forall n_1, n_2$$

Stationarity

A stochastic process $x(n)$ is called stationary of order N if

$$f_x(x_1, \ldots, x_N; n_1, \ldots, n_N) = f_x(x_1, \ldots, x_N; n_1 + k, \ldots, n_N + k)$$

for any value of k.

If $x(n)$ is stationary for all orders $N = 1, 2, \ldots$, it is said to be strict-sense stationary (SSS).

- An IID sequence is SSS.
- If a stochastic process $x(n)$ is stationary up to order 2, it is said to be wide-sense stationary (WSS).

Wide-Sense Stationary

A random signal x(n) is called wide-sense stationary (WSS) if

1. Its mean is a constant independent of n,
2. Its variance is also a constant independent of n,
3. Its autocorrelation depends only on the distance $l = n_1 - n_2$, called lag.

Wide-Sense Stationary

A random signal x(n) is called wide-sense stationary (WSS) if

1. $E\{x(n)\} = \mu_x$
2. $Var[x(n)] = \sigma_x^2$
3. $r_x(n_1, n_2) = r_x(n_1 - n_2) = r_x(l) = E\{x(n+l)x^*(n)\} = E\{x(n)x^*(n-l)\}$

Ergodicity

Can we infer the statistical characteristics of the process from a single realization?

Ergodicity implies that all the statistical information can be obtained from any single representative member of the ensemble.

Time average

Time average:

$$\langle(\cdot)\rangle = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} (\cdot)$$

- Mean value $= \langle x(n) \rangle$
- Mean square $= \langle |x(n)|^2 \rangle$
- Variance $= \langle |x(n) - \langle x(n) \rangle|^2 \rangle$
- Autocorrelation $= \langle x(n) x^*(n-l) \rangle$
- Autocovariance $= \langle [x(n) - \langle x(n) \rangle][x(n-l) - \langle x(n) \rangle]^* \rangle$
- Cross-correlation $= \langle x(n) y^*(n-l) \rangle$
- Cross-covariance $= \langle [x(n) - \langle x(n) \rangle][y(n-l) - \langle y(n) \rangle]^* \rangle$

Ergodic Random Processes

A random signal $x(n)$ is called ergodic if its ensemble averages equal appropriate time averages.

If the process is stationary and ergodic, then all statistical information can be derived from only one typical realization of the process.

Ergodic Random Processes

A random process $x(n)$ is ergodic in the mean if

$$\langle x(n) \rangle = E\{x(n)\}$$

A random process $x(n)$ is ergodic in correlation if

$$\langle x(n)x^*(n-l) \rangle = E\{x(n)x^*(n-l)\}$$

- That is, $\langle x(n) \rangle$ is a constant and $\langle x(n)x^*(n-l) \rangle$ is a function of l. If $x(n)$ is ergodic in both mean and variance, it is WSS. Only stationary signals can be ergodic.
- On the other hand, WSS does not imply ergodicity of any kind.

Frequency-Domain Description of Stationary Processes

- Power spectral density
- White noise
  A random sequence $w(n)$ is called a (second-order) white noise process with mean μ_w and variance σ_w^2, if and only if

$$E\{w(n)\} = \mu_w$$

and

$$r_w(l) = E\{w(n)w^*(n-l)\} = \sigma_w^2 \delta(l)$$

$$R_w(e^{j\omega}) = \sigma_w^2$$

Random Vectors

An M-dimensional real-valued random vector

$$\mathbf{x}(\zeta) = [x_1(\zeta), x_2(\zeta), \ldots, x_M(\zeta)]^T$$

is completely characterized by its joint cdf:

$$F_{\mathbf{x}}(x_1, \ldots, x_M) = Pr\{x_1(\zeta) \leq x_1, \ldots, x_M(\zeta) \leq x_M\} = Pr\{\mathbf{x}(\zeta) \leq \mathbf{x}\}$$

or by its joint pdf:

$$f_{\mathbf{x}}(\mathbf{x}) = \frac{\partial}{\partial x_1} \cdots \frac{\partial}{\partial x_M} F_{\mathbf{x}}(\mathbf{x})$$

Complex-valued Random Vectors

An M-dimensional complex-valued random vector

$$\mathbf{x}(\zeta) = \mathbf{x}_R(\zeta) + j\mathbf{x}_I(\zeta)$$

is completely characterized by its joint cdf:

$$F_{\mathbf{x}}(\mathbf{x}) = Pr\{\mathbf{x}(\zeta) \leq \mathbf{x}\} = Pr\{\mathbf{x}_R(\zeta) \leq \mathbf{x}_R, \mathbf{x}_I(\zeta) \leq \mathbf{x}_I\}$$

or by its joint pdf:

$$f_{\mathbf{x}}(\mathbf{x}) = \frac{\partial}{\partial x_{R1}} \frac{\partial}{\partial x_{I1}} \cdots \frac{\partial}{\partial x_{RM}} \frac{\partial}{\partial x_{IM}} F_{\mathbf{x}}(\mathbf{x})$$

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- If two random vectors $\mathbf{x}$ and $\mathbf{y}$ are orthogonal

$$\mathbf{R}_{\mathbf{xy}} = \mathbf{0}$$

General Correlation Matrices

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- The eigenvalues $\{\lambda_i\}_{i=1}^{M}$ of correlation matrix $\mathbf{R}_x$ are real and nonnegative
- If $\mathbf{R}$ is positive definite, then $\lambda_i > 0$ for all $1 \leq i \leq M$.
- If the eigenvalues $\{\lambda_i\}_{i=1}^{M}$ are distinct, then the corresponding eigenvectors are orthogonal to one another, that is,

$$\lambda_i \neq \lambda_j \Rightarrow \mathbf{q}_i^H \mathbf{q}_j = 0, \text{ for } \ i \neq j$$

Spectral Decomposition

Let $\{\mathbf{q}_i\}_{i=1}^{M}$ be an orthonormal set of eigenvectors corresponding to the distinct eigenvalues $\{\lambda_i\}_{i=1}^{M}$ of an $M \times M$ correlation matrix $\mathbf{R}$.

Then $\mathbf{R}$ can be diagonalized as

$$\boldsymbol{\Lambda} = \mathbf{Q}^H \mathbf{R} \mathbf{Q}$$

where the orthonormal matrix $\mathbf{Q} = [\mathbf{q}_1 \cdots \mathbf{q}_M]$ is known as an eigen-matrix and $\boldsymbol{\Lambda}$ is an $M \times M$ diagonal eigenvalue matrix, that is,

$$\boldsymbol{\Lambda} = \mathrm{diag}(\lambda_1, \ldots, \lambda_M).$$

The trace of $\mathbf{R}$ is the summation of all eigenvalues

$$\mathrm{tr}(\mathbf{R}) = \sum_{i=1}^{M} \lambda_i$$

Correlation Matrices from Random Processes

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- Correlation:

$$\mathbf{R}_x(n) = \begin{bmatrix} r_x(n,n) & \cdots & r_x(n, n-M+1) \\ \vdots & \ddots & \vdots \\ r_x(n-M+1, n) & \cdots & r_x(n-M+1, n-M+1) \end{bmatrix}$$

Correlation Matrices from Random Processes

- Correlation matrices of stationary processes $\mathbf{R}_x$

$$\mathbf{R}_x(n) = \begin{bmatrix} r_x(0) & r_x(1) & \cdots & r_x(M-1) \\ r_x^*(1) & r_x(0) & \cdots & r_x(M-2) \\ \vdots & \vdots & \ddots & \vdots \\ r_x^*(M-1) & r_x^*(M-2) & \cdots & r_x(0) \end{bmatrix}$$

- $\mathbf{R}_x$ is Hermitian and Toeplitz.

Whitening and Innovations Representation

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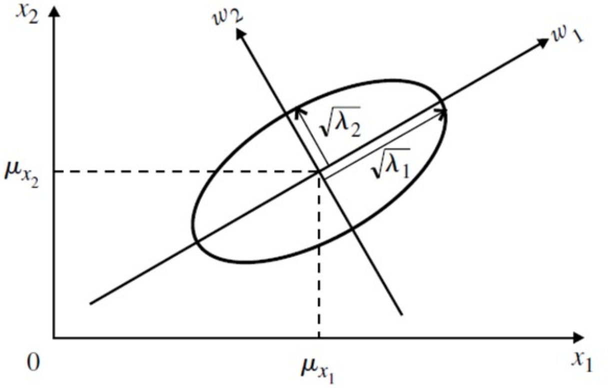
- Furthermore, if $\mathbf{w}$ has uncorrelated components and $\mathbf{A}$ is lower-triangular, then each component w_i of $\mathbf{w}$ can be thought of as adding "new" information (or innovation) to $\mathbf{w}$.

Transformations Using Eigen-decomposition

- Let $\mathbf{Q}_x$ be the eigenmatrix of $\mathbf{x}$. Then $\mathbf{Q}_x^H$ is the linear transformation matrix $\mathbf{A}$.
- The variances of random variables $w_i, i = 1, \ldots, M$, are equal to the eigenvalues of $\mathbf{x}$.
- Since the transformation matrix $\mathbf{A} = \mathbf{Q}_x^H$ is orthonormal, the transformation is called an orthonormal transformation.

Orthogonal Transformation



Isotropic Transformation

$$\mathbf{y} = \mathbf{\Lambda}_x^{-1/2}\mathbf{w} \qquad \mathbf{R}_y = \mathbf{I}$$