ELC 5396: Digital Communications

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September 22, 2016

Signaling over AWGN Channels

The 1s and 0s emitted by the communication source are encoded into distinct signals denoted by $s_1(t)$ and $s_2(t)$, respectively, which are suitable for transmission over the analog channel.

Symbols $s_1(t)$ and $s_2(t)$ are real-valued energy signals.

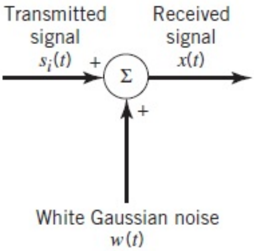
$$E_i = \int_0^{T_b} s_i^2(t)dt, \quad i = 1, 2$$

AWGN Channel

$$x(t) = s_i(t) + w(t), \quad i = 1, 2$$

where, $w(t)$ is the channel noise.

Signaling over AWGN Channels



Average probability of symbol error

$$P_e = \pi_1 \mathbb{P}(\hat{m} = 0|1 \text{ sent}) + \pi_2 \mathbb{P}(\hat{m} = 1|0 \text{ sent})$$

where π_1 and π_2 are the prior probabilities of transmitting symbols 1 and 0, respectively.

Geometric Representation of Signals

The essence of geometric representation of signals is to represent any set of M energy signals $\{s_i(t)\}$ as linear combinations of N orthonormal basis functions, where $N \leq M$.

$$s_i(t) = \sum_{j=1}^{N} s_{ij}\phi_j(t), \quad i = 1, 2, \ldots, M$$

where,

$$s_{ij} = \int_0^T s_i(t)\phi_j(t)dt, \quad i = 1, 2, \ldots, M, \quad j = 1, 2, \ldots, N$$

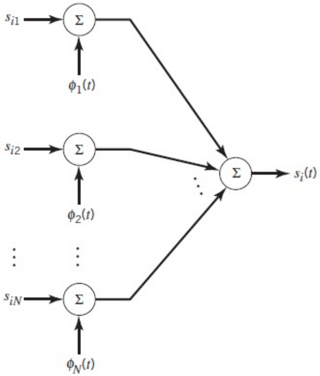
Geometric Representation of Signals

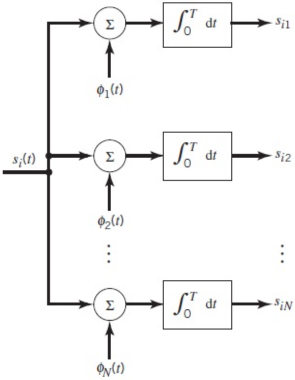
The real-valued basis functions $\phi_i(t)$ form an orthonormal set

$$\int_0^T \phi_i(t)\phi_j(t)dt = \delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

Geometric Representation of Signals

A synthesizer and an analyzer:



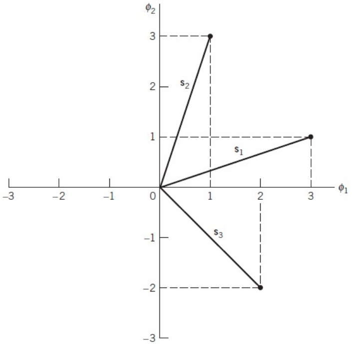


Geometric Representation of Signals

Accordingly, we may state that each signal in the set $\{s_i(t)\}$ is completely determined by the signal vector

$$\mathbf{s}_i = [s_{i1}, s_{i2}, \cdots, s_{iN}]^T, \quad i = 1, 2 \ldots, M$$

$\mathbf{s}_i$ is a point in an N-dimensional Euclidean space which is called the signal space.



Geometric Representation of Signals

Length of a signal vector – norm: $\|\mathbf{s}_i\|$

$$\|\mathbf{s}_i\|^2 = \mathbf{s}_i^H \mathbf{s}_i = \sum_{j=1}^{N} |s_{ij}|^2$$

The energy of a signal:

$$E_i = \int_0^T |s_i|^2 dt = \sum_{j=1}^{N} |s_{ij}|^2 = \|\mathbf{s}_i\|^2$$

The inner product of the energy signals $s_i(t)$ and $s_k(t)$ over the interval $[0, T]$ is equal to the inner product of their respective vector representations $\mathbf{s}_i$ and $\mathbf{s}_k$:

$$\int_0^T s_i^*(t) s_k(t) dt = \mathbf{s}_i^H \mathbf{s}_k$$

Geometric Representation of Signals

Euclidean distance between two signal vectors:

$$\|\mathbf{s}_i - \mathbf{s}_k\|^2 = \sum_{j=1}^{N} (s_{ij} - s_{kj})^2$$

The angle between two signal vectors:

$$\cos(\theta_{ik}) = \frac{\mathbf{s}_i^H \mathbf{s}_k}{\|\mathbf{s}_i\| \|\mathbf{s}_k\|}$$

Examples to read: (1) The Schwarz Inequality, (2) Gram-Schmidt Orthogonalization Procedure

Conversion of the Continuous AWGN Channel into a Vector Channel

The received signal over AWGN channel:

$$x(t) = s_i(t) + w(t), \;\; i = 1, 2, \ldots, M$$

Using the "analyzer", the output of the jth correlater:

$$x_j = \int_0^T x(t)\phi_j(t)dt = s_{ij} + w_j, \;\; j = 1, 2, \ldots, N$$

where $w_j = \int_0^T w(t)\phi_j(t)dt$.

The received signal can be expressed as

$$x(t) = \sum_{j=1}^{N} x_j \phi_j(t) + w'(t)$$

where $w'(t) = w(t) - \sum_{j=1}^{N} w_j \phi_j(t)$.

Statistical Characterization of the Correlator Outputs

The output of the jth correlator is a Gaussian random variable X_j. $(j = 1, 2, \ldots, N)$

$$\mu_{X_j} = \mathbb{E}[s_{ij} + W_j] = s_{ij} + \mathbb{E}[W_j] = s_{ij}$$

$$\sigma_{X_j}^2 = var[X_j] = \mathbb{E}[W_j^2] = \frac{N_0}{2}, \ \forall j$$

where

$$W_j = \int_0^T W(t)\phi_j(t)dt$$

$$cov[X_j X_k] = 0, \ j \neq k$$

Statistical Characterization of the Correlator Outputs

Define the vector of N random variables

$$\mathbf{X} = [X_1, X_2, \cdots, X_N]^T$$

whose elements are independent Gaussian random variables with mean values equal to s_{ij} and variances equal to $N_0/2$.

Conditional pdf of the vector $\mathbf{X}$

$$f_{\mathbf{X}}(\mathbf{x}|m_i) = \prod_{j=1}^{N} f_{X_j}(x_j|m_i), \quad i = 1, 2, \ldots, M$$

where $\mathbf{x}$ is the observation vector and x_j is an element of the observation vector.

Statistical Characterization of the Correlator Outputs

Since each X_j is a Gaussian random variable with mean s_{ij} and variance $N_0/2$, we have

$$f_{X_j}(x_j|m_i) = \frac{1}{\sqrt{\pi N_0}} \exp\left(-\frac{1}{N_0}(x_j - s_{ij})^2\right)$$

Therefore,

$$f_{\mathbf{X}}(\mathbf{x}|m_i) = (\pi N_0)^{-N/2} \exp\left(-\frac{1}{N_0}\sum_{j=1}^{N}(x_j - s_{ij})^2\right)$$

Statistical Characterization of the Correlator Outputs

The remainder of the noise is irrelevant.

$$\mathbb{E}[X_j W'] = 0, \;\; j = 1, 2, \ldots, N$$

Statistical Characterization of the Correlator Outputs

The AWGN channel model is an N-dimensional vector channel

$$\mathbf{x} = \mathbf{s}_i + \mathbf{w}, \quad i = 1, 2, \ldots, M$$

where the dimension N is the number of basis functions involved in formulating the signal vector $\mathbf{s}_i$ for all i.

Likelihood Function

Likelihood function:

$$l(m_i|\mathbf{x}) = f_{\mathbf{X}}(\mathbf{x}|m_i)$$

The likelihood $l(m_i|\mathbf{x})$ is not a distribution; rather, it is a function of the parameter m_i, given $\mathbf{x}$.

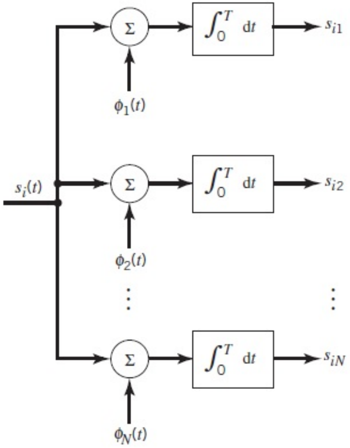
The log-likelihood function for an AWGN channel is

$$L(m_i) = \ln l(m_i) = -\frac{1}{N_0}\sum_{j=1}^{N}(x_j - s_{ij})^2, \quad i = 1, 2, \ldots, M$$

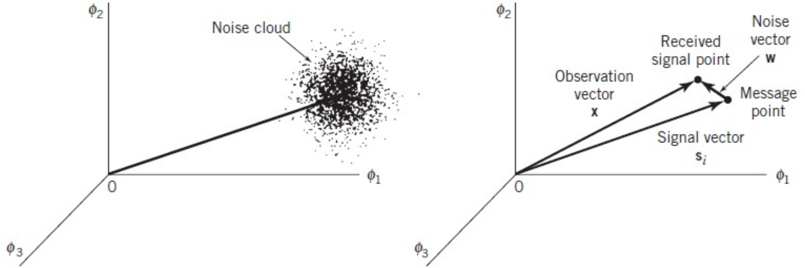
where we have ignored the constant term $-(N/2)\ln(\pi N_0)$ since it bears no relation to the message symbol m_i.
Recall that $s_{ij}, j = 1, 2, \ldots, N$, are the elements of the signal vector $\mathbf{s}_i$ representing the message symbol m_i.

Maximum Likelihood Decoding

A bank of correlators:

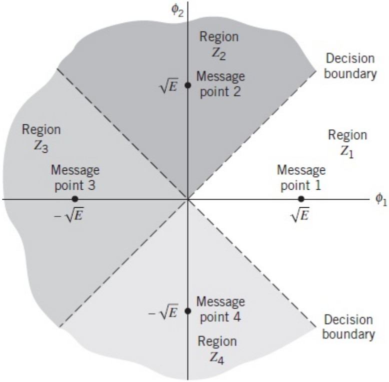


Optimum Receivers Using Coherent Detection



$$\mathbf{x} = \mathbf{s}_i + \mathbf{w}$$

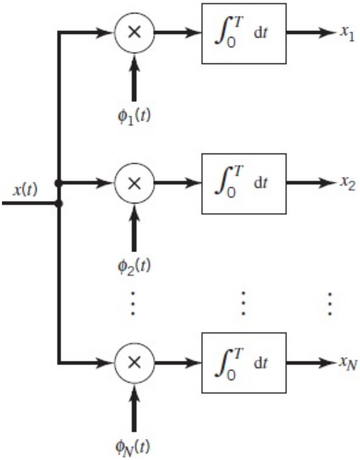
Optimum Receivers Using Coherent Detection



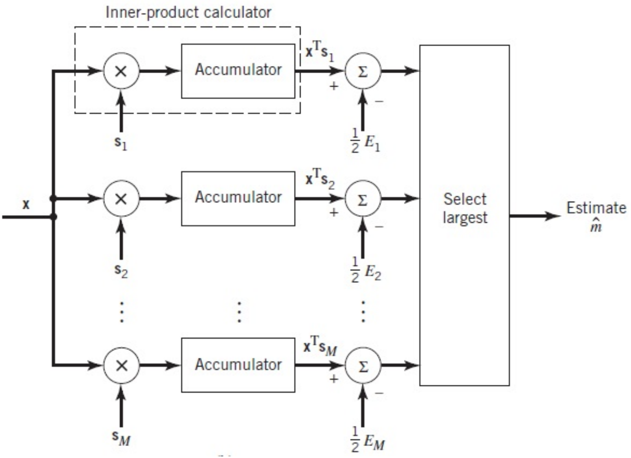
Observation vector **x** lies in region Z_i if $(\sum_{j=1}^{N} x_j s_{kj} - \frac{1}{2} E_k)$ is maximum for $k = i$, where E_k is transmitted energy.

Correlation Receiver

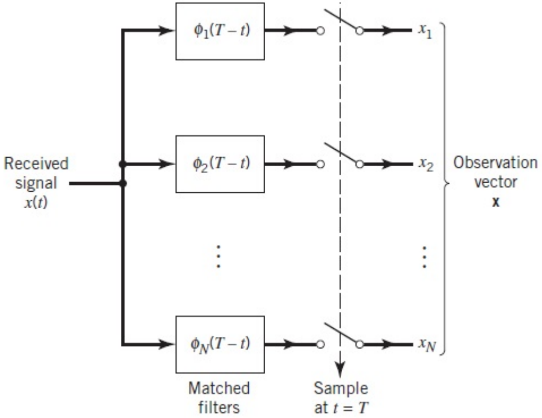
(a) Detector



(b) Maximum-likelihood Decoder



Matched Filter Receiver



Probability of Error

$$P_e = 1 - \frac{1}{M} \sum_{i=1}^{M} \int_{Z_i} f_{\mathbf{X}}(\mathbf{x}|m_i) d\mathbf{x}$$

Probability of Error

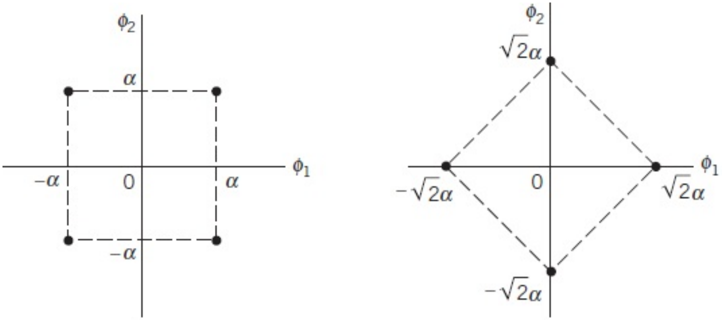
- Invariance of the Probability of Error to Rotation

  If a message constellation is rotated by the transformation

  $$\mathbf{s}_{i,rotate} = \mathbf{Q}\mathbf{s}_i, \quad i = 1, 2, \ldots, M$$

  where $\mathbf{Q}$ is an orthonormal matrix, then the probability of symbol error P_e incurred in maximum likelihood signal-detection over an AWGN channel is completely unchanged.

Illustration of Rotational Invariance



Probability of Error

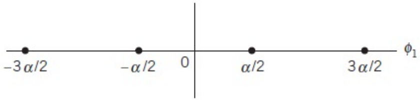
- Invariance of the Probability to Translation

  If a signal constellation is translated by a constant vector amount,

  $$\mathbf{s}_{i,translate} = \mathbf{s}_i - \mathbf{a}, \quad i = 1, 2, \dots, M$$

  then the probability of symbol error P_e incurred in maximum likelihood signal detection over an AWGN channel is completely unchanged.

Illustration of Translation Invariance



Union Bound on the Probability of Error

Union Bound

$$P_e(m_i) \leq \sum_{k=1, k\neq i}^{M} \mathbb{P}(A_{ik}), \quad i = 1, 2, \ldots, M$$

Pairwise Error Probability

$$P_e(m_i) \leq \sum_{k=1, k\neq i}^{M} p_{ik}, \quad i = 1, 2, \ldots, M$$

where p_{ik} is the pairwise error probability.