Sampling Theory

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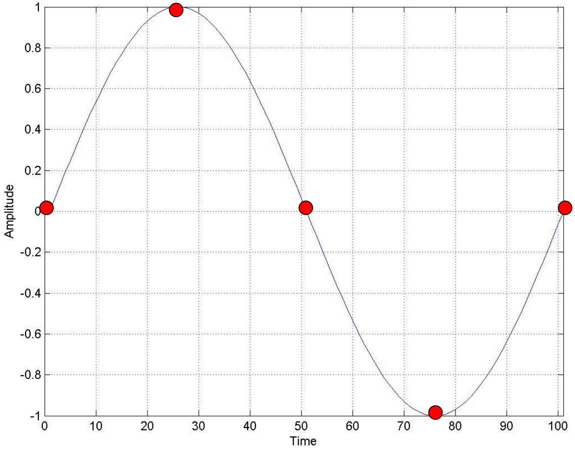
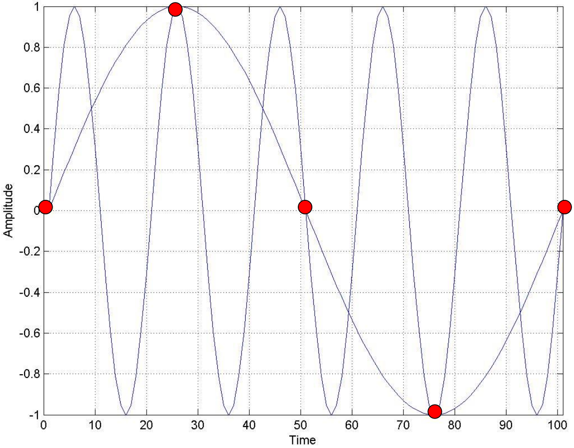
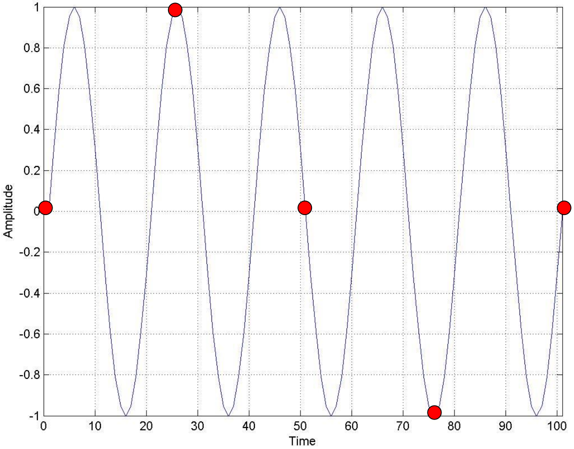
Sampling Dilemma



Sampling Dilemma



Sampling Dilemma



The Theory

Sampling Theorem

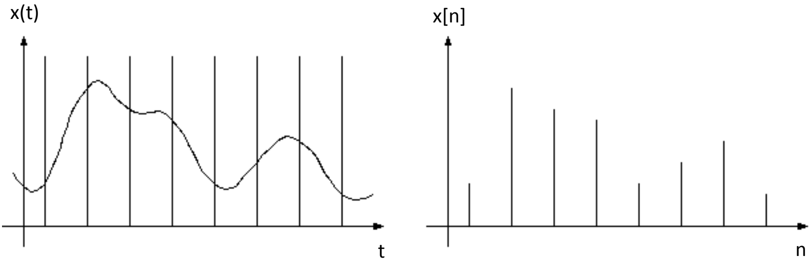
If a signal $x(t)$ contains no frequency components for frequencies above $f = W$ hertz, then it is completely described by instantaneous sample values uniformly spaced in time with period $T_s \leq 1/2W$.

That is, the sampling frequency $f_s = 1/T_s$ needs to satisfy

$$f_s \geq 2W.$$

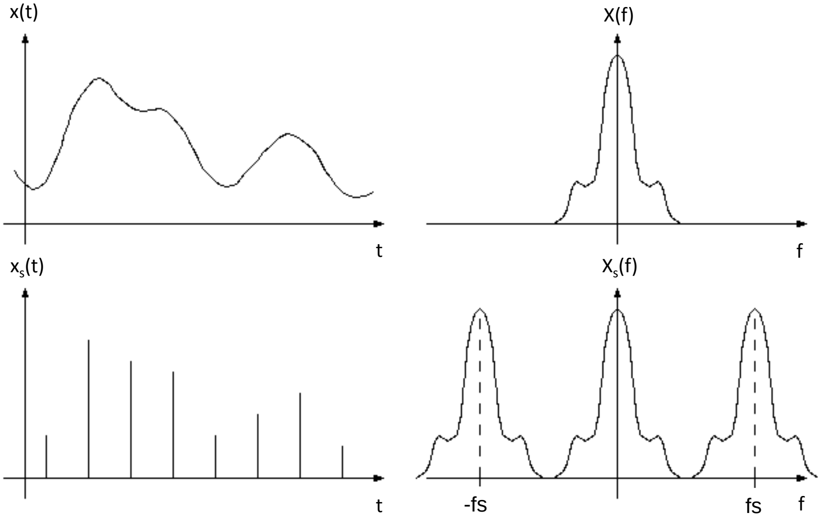
The frequency $2W$ is referred to as the *Nyquist frequency*.

The Proof



Suppose that $x(t)$ is a continuous-time signal. $x[n]$ is the discrete-time signal that consists of samples of $x(t)$ with a sampling period T_s. Therefore,

$$x[n] = x(nT_s), \qquad -\infty < n < \infty.$$


x(t)
t
X(f)
f
$x_s(t)$
t
$X_s(f)$
-fs
fs
f

- Suppose that the Fourier transform of $x(t)$ is $X(f)$. That is,

$$X(f) = \mathcal{F}\{x(t)\}.$$

- The continuous-time representation of the sampled signal is

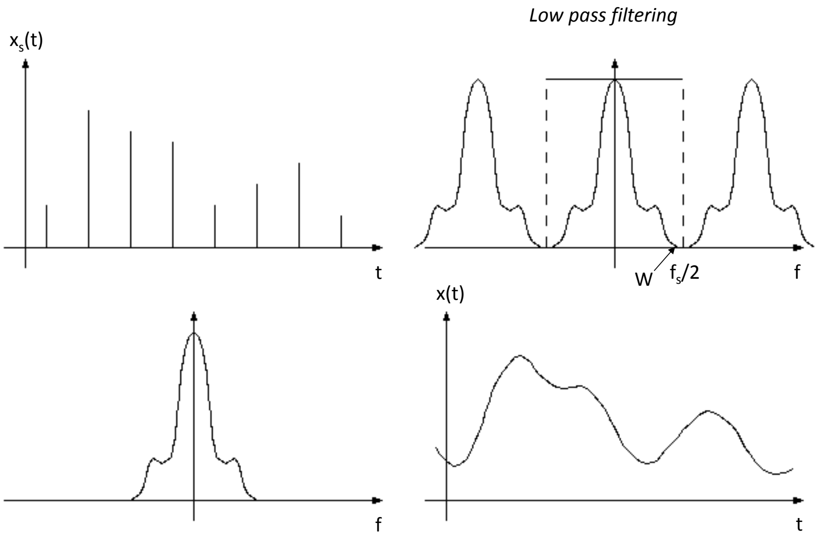
$$\begin{aligned} x_s(t) &= \sum_{n=-\infty}^{\infty} x(nT_s)\delta(t - nT_s) \\ &= x(t) \sum_{n=-\infty}^{\infty} \delta(t - nT_s) \end{aligned}$$

where $\delta(t)$ is the Dirac delta function.

- The Fourier transform of $x_s(t)$ is $X_s(f)$, which can be calculated as

$$\begin{aligned} X_s(f) = \mathcal{F}\{x_s(t)\} &= X(f) \otimes \mathcal{F}\left\{ \sum_{n=-\infty}^{\infty} \delta(t - nT_s) \right\} \\ &= X(f) \otimes \left[f_s \sum_{n=-\infty}^{\infty} \delta(f - nf_s) \right] \\ &= f_s X(f) \otimes \sum_{n=-\infty}^{\infty} \delta(f - nf_s) \\ &= f_s \sum_{n=-\infty}^{\infty} X(f) \otimes \delta(f - nf_s) \\ &= f_s \sum_{n=-\infty}^{\infty} X(f - nf_s) \end{aligned}$$

where, $f_s = 1/T_s$ is the sampling frequency.


Low pass filtering
xs(t)
t
W
fs/2
f
x(t)
f
t

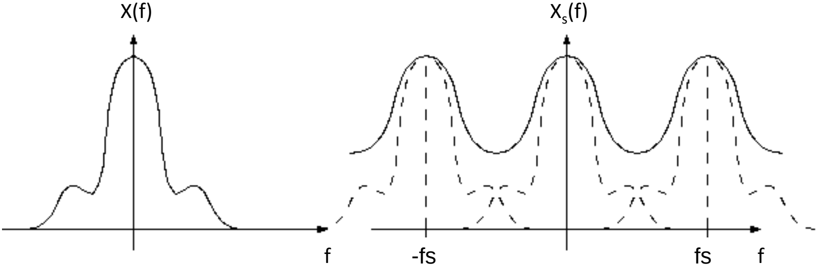
- In order to reconstruct the original signal $x(t)$, we need to pass the sampled signal $x_s(t)$ through an ideal low-pass filter (rectangular function in frequency) to remove the high-frequency replicas.

- A prefect $X(f)$ can be extracted by applying the rectangular function for filtering only when

$$f_s/2 \geq W$$

where W is the largest frequency component in signal $x(t)$. $\square$

Aliasing



Sampling rate f_s is smaller than the Nyquist rate $2W$.