

# ELC 4351: Digital Signal Processing

Liang Dong

Electrical and Computer Engineering  
Baylor University

*liang.dong@baylor.edu*

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# Frequency Analysis of Signals

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  - Frequency-Domain and Time-Domain Signal Properties
- 2 Properties of the Fourier Transform for Discrete-Time Signals
  - Symmetry Properties of the Fourier Transform
  - Fourier Transform Theorems and Properties

## Frequency Analysis Tools

The Fourier series	for continuous-time periodic signals
The Fourier transform	for continuous-time aperiodic signals
The Fourier series	for discrete-time periodic signals
The Fourier transform	for discrete-time aperiodic signals

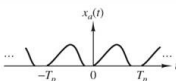
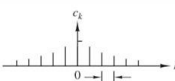
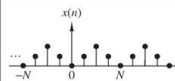
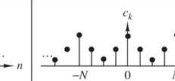
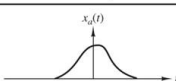
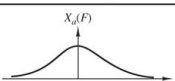
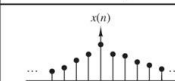
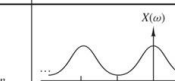
Continuous-time signals have aperiodic spectra

Discrete-time signals have periodic spectra

Periodic signals have discrete spectra

Aperiodic finite energy signals have continuous spectra

# The Fourier Series for Continuous-Time Periodic Signals

		Continuous-time signals		Discrete-time signals	
		Time-domain	Frequency-domain	Time-domain	Frequency-domain
Periodic signals Fourier series		 $c_k = \frac{1}{T_p} \int_{T_p} x_a(t) e^{-j2\pi k F t} dt$	 $F_0 = \frac{1}{T_p}$	 $c_k = \frac{1}{N} \sum_{n=0}^{N-1} x(n) e^{-j2\pi/N kn}$	 $x(n) = \sum_{k=0}^{N-1} c_k e^{j2\pi/N kn}$
		Continuous and periodic	Discrete and aperiodic	Discrete and periodic	Discrete and periodic
Aperiodic signals Fourier transforms		 $X_a(F) = \int_{-\infty}^{\infty} x_a(t) e^{-j2\pi F t} dt$	 $x_a(t) = \int_{-\infty}^{\infty} X_a(F) e^{j2\pi F t} dF$	 $X(\omega) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n}$	 $x(n) = \frac{1}{2\pi} \int_{2\pi} X(\omega) e^{j\omega n} d\omega$
		Continuous and aperiodic	Continuous and aperiodic	Discrete and aperiodic	Continuous and periodic

Periodicity with period  $\alpha$  in one domain implies discretization with spacing  $1/\alpha$  in the other domain, and *vice versa*.

$$X(\omega) \triangleq F\{x(n)\} = \sum_{n=-\infty}^{\infty} x(n)e^{-j\omega n}$$

$$x(n) \triangleq F^{-1}\{X(\omega)\} = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\omega)e^{j\omega n} d\omega$$

Fourier transform pair:  $x(n) \longleftrightarrow^F X(\omega)$

where,  $X(\omega)$  is periodic with period  $2\pi$ .

If signal is complex, it can be expressed in rectangular form

$$x(n) = x_R(n) + jx_I(n)$$

$$X(\omega) = X_R(\omega) + jX_I(\omega)$$

# Symmetry Properties of the Fourier Transform

When a signal satisfies some symmetry properties in the time domain, these properties impose some symmetry conditions on its Fourier transform.

Using the rectangular form and  $e^{j\omega} = \cos \omega + j \sin \omega$ , we have

$$X_R(\omega) = \sum_{n=-\infty}^{\infty} [x_R(n) \cos \omega n + x_I(n) \sin \omega n]$$

$$X_I(\omega) = - \sum_{n=-\infty}^{\infty} [x_R(n) \sin \omega n - x_I(n) \cos \omega n]$$

and

$$x_R(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} [X_R(\omega) \cos \omega n - X_I(\omega) \sin \omega n] d\omega$$

$$x_I(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} [X_R(\omega) \sin \omega n + X_I(\omega) \cos \omega n] d\omega$$

# Symmetry Properties of the Fourier Transform

Real signals.  $x_R(n) = x(n)$  and  $x_I(n) = 0$ .

$$X_R(\omega) = \sum_{n=-\infty}^{\infty} x(n) \cos \omega n$$

$$X_I(\omega) = - \sum_{n=-\infty}^{\infty} x(n) \sin \omega n$$

It follows that

$$X_R(-\omega) = X_R(\omega)$$

$$X_I(-\omega) = -X_I(\omega)$$

$\implies X^*(\omega) = X(-\omega)$ . The spectrum of a real signal has *Hermitian symmetry*.

# Symmetry Properties of the Fourier Transform

Real signals.  $x_R(n) = x(n)$  and  $x_I(n) = 0$ .

$$X_R(-\omega) = X_R(\omega) \quad (\text{even})$$

$$X_I(-\omega) = -X_I(\omega) \quad (\text{odd})$$

$$|X(-\omega)| = |X(\omega)| \quad (\text{even})$$

$$\angle X(-\omega) = -\angle X(\omega) \quad (\text{odd})$$



# Symmetry Properties of the Fourier Transform

Real and even signals.  $x_R(n) = x(n)$ ,  $x_I(n) = 0$  and  $x(-n) = x(n)$ .

$$X_R(\omega) = x(0) + 2 \sum_{n=1}^{\infty} x(n) \cos \omega n \quad (\text{even})$$

$$X_I(\omega) = 0$$

It has real-valued spectrum, which is even function of the frequency  $\omega$ .

# Symmetry Properties of the Fourier Transform

Real and odd signals.  $x_R(n) = x(n)$ ,  $x_I(n) = 0$  and  $x(-n) = -x(n)$ .

$$X_R(\omega) = 0$$

$$X_I(\omega) = -2 \sum_{n=1}^{\infty} x(n) \sin \omega n \quad (\text{odd})$$

It has imaginary-valued spectrum, which is odd function of the frequency  $\omega$ .

# Symmetry Properties of the Fourier Transform

Purely imaginary signals.  $x_R(n) = 0$  and  $jx_I(n) = x(n)$ .

$$X_R(\omega) = \sum_{n=-\infty}^{\infty} x_I(n) \sin \omega n \quad (\text{odd})$$

$$X_I(\omega) = \sum_{n=-\infty}^{\infty} x_I(n) \cos \omega n \quad (\text{even})$$

# Symmetry Properties of the Fourier Transform

Purely imaginary and odd signals.  $x_R(n) = 0$ ,  $jx_I(n) = x(n)$  and  $x_I(-n) = -x_I(n)$ .

$$X_R(\omega) = 2 \sum_{n=1}^{\infty} x_I(n) \sin \omega n \quad (\text{odd})$$
$$X_I(\omega) = 0$$

It has real-valued spectrum, which is odd function of the frequency  $\omega$ .

# Symmetry Properties of the Fourier Transform

Purely imaginary and even signals.  $x_R(n) = 0$ ,  $jx_I(n) = x(n)$  and  $x_I(-n) = x_I(n)$ .

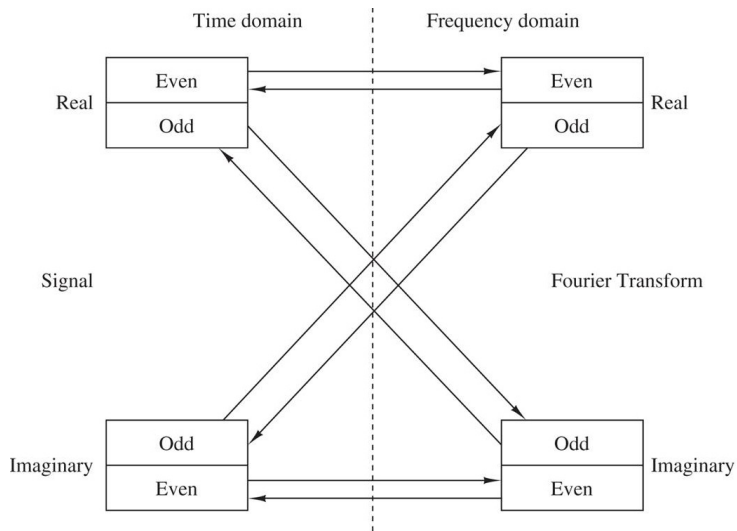
$$X_R(\omega) = 0$$

$$X_I(\omega) = x_I(0) + 2 \sum_{n=1}^{\infty} x_I(n) \cos \omega n \quad (\text{even})$$

It has imaginary-valued spectrum, which is even function of the frequency  $\omega$ .

# Symmetry Properties of the Fourier Transform

## Summary of symmetry properties for the Fourier Transform



# Fourier Transform Theorems and Properties

Linearity.

If  $x_1(n) \longleftrightarrow X_1(\omega)$  and  $x_2(n) \longleftrightarrow X_2(\omega)$ ,

then

$$\alpha_1 x_1(n) + \alpha_2 x_2(n) \longleftrightarrow \alpha_1 X_1(\omega) + \alpha_2 X_2(\omega).$$

# Fourier Transform Theorems and Properties

Time shifting.

If  $x(n) \longleftrightarrow X(\omega)$ ,

then

$x(n - k) \longleftrightarrow e^{-j\omega k} X(\omega)$ .



# Fourier Transform Theorems and Properties

Time reversal.

If  $x(n) \longleftrightarrow X(\omega)$ ,

then

$x(-n) \longleftrightarrow X(-\omega)$ .

# Fourier Transform Theorems and Properties

Convolution theorem.

If  $x_1(n) \longleftrightarrow X_1(\omega)$  and  $x_2(n) \longleftrightarrow X_2(\omega)$ ,  
then

$$x(n) = x_1(n) \otimes x_2(n) \longleftrightarrow X(\omega) = X_1(\omega)X_2(\omega).$$

Correlation theorem.

If  $x_1(n) \longleftrightarrow X_1(\omega)$  and  $x_2(n) \longleftrightarrow X_2(\omega)$ ,  
then

$$r_{x_1x_2}(l) \longleftrightarrow S_{x_1x_2}(\omega) = X_1(\omega)X_2(-\omega).$$

# Fourier Transform Theorems and Properties

The Wiener-Khintchine theorem.

If  $x(n)$  is a real signal, then  $r_{xx}(l) \longleftrightarrow S_{xx}(\omega)$ .

Notice that neither the autocorrelation nor the energy spectral density has any phase information.

# Fourier Transform Theorems and Properties

Frequency shifting.

If  $x(n) \longleftrightarrow X(\omega)$ ,

then

$e^{j\omega_0 n} x(n) \longleftrightarrow X(\omega - \omega_0)$ .

# Fourier Transform Theorems and Properties

The modulation theorem.

If  $x(n) \longleftrightarrow X(\omega)$ ,

then

$$x(n) \cos \omega_0 n \longleftrightarrow \frac{1}{2}[X(\omega + \omega_0) + X(\omega - \omega_0)].$$

# Fourier Transform Theorems and Properties

Parseval's theorem.

If  $x_1(n) \longleftrightarrow X_1(\omega)$  and  $x_2(n) \longleftrightarrow X_2(\omega)$ ,  
then

$$\sum_{n=-\infty}^{\infty} x_1(n)x_2^*(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} X_1(\omega)X_2^*(\omega)d\omega.$$

# Fourier Transform Theorems and Properties

Windowing theorem.

If  $x_1(n) \longleftrightarrow X_1(\omega)$  and  $x_2(n) \longleftrightarrow X_2(\omega)$ ,  
then

$$x_1(n)x_2(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} X_1(\lambda)X_2(\omega - \lambda)d\lambda.$$



# Fourier Transform Theorems and Properties

Differentiation in the frequency domain.

If  $x(n) \longleftrightarrow X(\omega)$ ,  
then

$$nx(n) \longleftrightarrow j \frac{dX(\omega)}{d\omega}.$$