ELC 4351: Digital Signal Processing

Liang Dong

Electrical and Computer Engineering
Baylor University

liang_dong@baylor.edu

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The z-Transform and Its Application to the Analysis of LTI Systems

Rational z-Transforms

$X(z)$ is a rational function, that is, a ratio of two polynomials in z^{-1} (or z).

$$
\begin{aligned}
X(z) &= \frac{B(z)}{A(z)} \\
&= \frac{b_0 + b_1 z^{-1} + \cdots + b_M z^{-M}}{a_0 + a_1 z^{-1} + \cdots a_N z^{-N}} \\
&= \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}}
\end{aligned}
$$

Rational z-Transforms

$X(z)$ is a rational function, that is, a ratio of two polynomials $B(z)$ and $A(z)$. The polynomials can be expressed in factored forms.

$$
\begin{aligned}
X(z) &= \frac{B(z)}{A(z)} \\
&= \frac{b_0}{a_0} z^{-M+N} \frac{(z - z_1)(z - z_2)\cdots(z - z_M)}{(z - p_1)(z - p_2)\cdots(z - p_N)} \\
&= \frac{b_0}{a_0} z^{N-M} \frac{\prod_{k=1}^{M}(z - z_k)}{\prod_{k=1}^{N}(z - p_k)}
\end{aligned}
$$

Poles and Zeros

The zeros of a z-transform $X(z)$ are the vales of z for which $X(z) = 0$.
The poles of a z-transform $X(z)$ are the vales of z for which $X(z) = \infty$.

$$X(z) = \frac{b_0}{a_0} z^{N-M} \frac{\prod_{k=1}^{M}(z - z_k)}{\prod_{k=1}^{N}(z - p_k)}$$

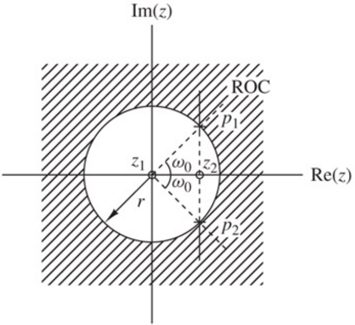
$X(z)$ has M finite zeros at $z = z_1, z_2, \ldots, z_M$, N finite poles at $z = p_1, p_2, \ldots, p_N$, and $|N - M|$ zeros (if $N > M$) or poles (if $N < M$) at the origin.

Poles and zeros may also occur at $z = \infty$.

$X(z)$ has exactly the same number of poles and zeros.

Poles and Zeros

If a polynomial has real coefficients, its roots are either real or occur in complex-conjugate pairs. That is because e.g. $(z - p_1)(z - p_2)$

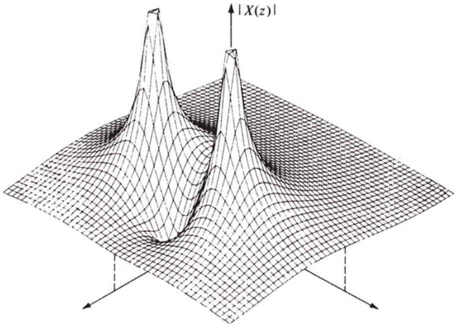


Poles and Zeros

For example,

$$X(z) = \frac{z^{-1} - z^{-2}}{1 - 1.2732z^{-1} + 0.81z^{-2}}$$

which has one zero at $z = 1$ and two poles at $p_1 = 0.9e^{j\pi/4}$ and $p_2 = 0.9e^{-j\pi/4}$.



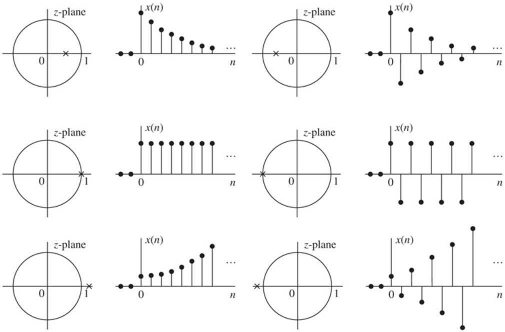
Some Common z-Transform Pairs

	Signal, $x(n)$	z-Transform, $X(z)$	ROC
1	$\delta(n)$	1	All z
2	$u(n)$	$\frac{1}{1-z^{-1}}$	$\|z\| > 1$
3	$a^n u(n)$	$\frac{1}{1-az^{-1}}$	$\|z\| > \|a\|$
4	$na^n u(n)$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$\|z\| > \|a\|$
5	$-a^n u(-n-1)$	$\frac{1}{1-az^{-1}}$	$\|z\| < \|a\|$
6	$-na^n u(-n-1)$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$\|z\| < \|a\|$
7	$(\cos\omega_0 n)u(n)$	$\frac{1-z^{-1}\cos\omega_0}{1-2z^{-1}\cos\omega_0+z^{-2}}$	$\|z\| > 1$
8	$(\sin\omega_0 n)u(n)$	$\frac{z^{-1}\sin\omega_0}{1-2z^{-1}\cos\omega_0+z^{-2}}$	$\|z\| > 1$
9	$(a^n\cos\omega_0 n)u(n)$	$\frac{1-az^{-1}\cos\omega_0}{1-2az^{-1}\cos\omega_0+a^2z^{-2}}$	$\|z\| > \|a\|$
10	$(a^n\sin\omega_0 n)u(n)$	$\frac{az^{-1}\sin\omega_0}{1-2az^{-1}\cos\omega_0+a^2z^{-2}}$	$\|z\| > \|a\|$

Poles Locations and Time-Domain Behavior for Causal Signals

If a real signal has a z-transform with one pole, this pole has to be real. The only such signal is the real exponential

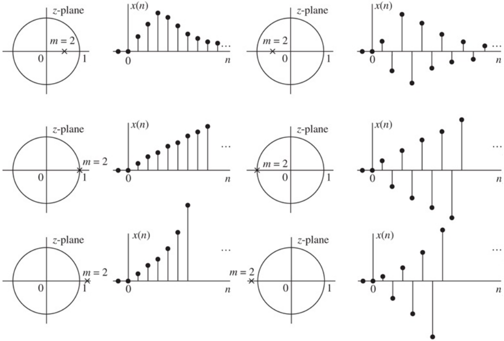
$$x(n) = a^n u(n) \rightarrow^z X(z) = \frac{1}{1 - az^{-1}}, \ \mathrm{ROC}: |z| > |a|$$



Poles Locations and Time-Domain Behavior for Causal Signals

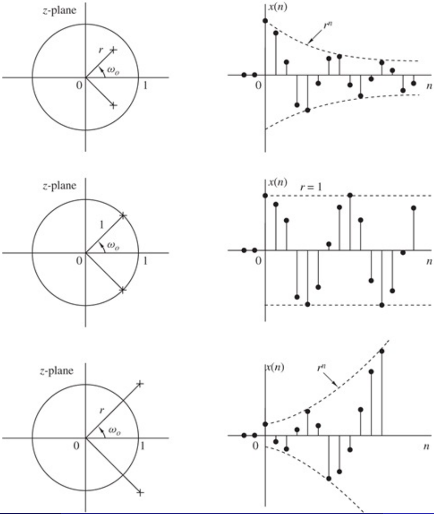
A causal real signal with a double real pole has the form

$$x(n) = na^n u(n) \xrightarrow{z} X(z) = \frac{az^{-1}}{(1 - az^{-1})^2}, \text{ ROC} : |z| > |a|$$



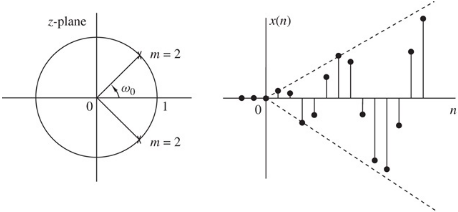
Poles Locations and Time-Domain Behavior for Causal Signals

The case of a causal signal with a pair of complex-conjugate poles.



Poles Locations and Time-Domain Behavior for Causal Signals

The case of a causal signal with a double pair of poles on the unit circle.



Poles Locations and Time-Domain Behavior for Causal Signals

The impulse response $h(n)$ of a causal LTI system is a causal signal.

Therefore, if a pole of a system is outside the unit circle, the impulse response of the system becomes unbounded and, consequently, the system is unstable.

System Function of a LTI System

LTI systems:

$$
\begin{aligned}
y(n) &= h(n) \otimes x(n) \\
Y(z) &= H(z)X(z)
\end{aligned}
$$

If we know the input $x(n)$ and observe the output $y(n)$ of the system, we can determine the unit sample response (impulse response) by first solving for $H(z)$ from

$$H(z) = \frac{Y(z)}{X(z)}$$

and then evaluating the inverse z-transform of $H(z)$.

$H(z)$ is called the system function.

System Function of a LTI System

When the LTI system is described by a linear constant-coefficient difference equation

$$y(n) = -\sum_{k=1}^{N} a_k y(n-k) + \sum_{k=0}^{M} b_k x(n-k)$$

The system function can be calculate:

$$\begin{aligned} Y(z) &= -\sum_{k=1}^{N} a_k Y(z) z^{-k} + \sum_{k=0}^{M} b_k X(z) z^{-k} \\ Y(z)\left(1 + \sum_{k=1}^{N} a_k z^{-k}\right) &= X(z)\left(\sum_{k=0}^{M} b_k z^{-k}\right) \\ H(z) &= \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{1 + \sum_{k=1}^{N} a_k z^{-k}} \end{aligned}$$

System Function of a LTI System

An LTI system described by a constant-coefficient difference equation has a rational system function $H(z)$.

$$\boxed{H(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{1 + \sum_{k=1}^{N} a_k z^{-k}}}$$

System Function of a LTI System

(1) All-zero system: If $a_k = 0$ for $1 \leq k \leq N$,

$$H(z) = \sum_{k=0}^{M} b_k z^{-k} = \frac{1}{z^M} \sum_{k=0}^{M} b_k z^{M-k}$$

The system has M nontrivial zeros and M trivial poles (at $z = 0$).

An all-zero system is an FIR system and can be called a moving average (MA) system.

System Function of a LTI System

(2) All-pole system: If $b_k = 0$ for $1 \le k \le M$,

$$H(z) = \frac{b_0}{1 + \sum_{k=1}^{N} a_k z^{-k}} = \frac{b_0 z^N}{\sum_{k=0}^{M} a_k z^{N-k}}$$

where $a_0 = 1$. The system has N nontrivial poles and N trivial zeros (at $z = 0$).

An all-pole system is an IIR system and can be called an auto-regressive (AR) system.

System Function of a LTI System

(3) Pole-zero system:

In general, the system function contains N poles and M zeros. (Poles and zeros at $z = 0$ and $z = \infty$ are implied but are not counted explicitly.)

Due to the presence of poles, a pole-zero system is an IIR system.

Inversion of the z-Transform

$$H(z) = \frac{Y(z)}{X(z)}, \qquad H(z) \rightarrow^{inv\ z} h(n)$$

Inverse z-Transform:

$$x(n) = \frac{1}{2\pi j} \oint_C X(z) z^{n-1} dz$$

where the integral is a (counter-clockwise) contour integral over a closed path C that encloses the origin and lies within the region of convergence of $X(z)$.

Methods of Inverse z-Transform

(1) Contour integration

(2) Power series expansion (using long division)

(3) Partial-fraction expansion

Inverse z-Transform by Partial-Fraction Expansion

$X(z)$ is rational function.

$$X(z) = \frac{B(z)}{A(z)} = \frac{b_0 + b_1 z^{-1} + \cdots + b_M z^{-M}}{1 + a_1 z^{-1} + \cdots + a_N z^{-N}}$$

A rational function is *proper* if $a_N \neq 0$ and $M < N$.

Inverse z-Transform by Partial-Fraction Expansion

An improper rational function ($M \geq N$) can always be written as the sum of a polynomial and a proper rational function.

$$X(z) = \frac{B(z)}{A(z)} = c_0 + c_1 z^{-1} + \cdots + c_{M-N} z^{-(M-N)} + \frac{B_1(z)}{A(z)}$$

The inverse z-transform of the polynomial can easily be found by inspection.

We focus our attention on the inversion of proper rational function.

Inverse z-Transform by Partial-Fraction Expansion

Let $X(z)$ be a proper rational function.

$$\begin{aligned} X(z) &= \frac{B(z)}{A(z)} = \frac{b_0 + b_1 z^{-1} + \cdots + b_M z^{-M}}{1 + a_1 z^{-1} + \cdots + a_N z^{-N}} \\ &= \frac{b_0 z^N + b_1 z^{N-1} + \cdots + b_M z^{N-M}}{z^N + a_1 z^{N-1} + \cdots + a_N} \end{aligned}$$

Since $N > M$,

$$\frac{X(z)}{z} = \frac{b_0 z^{N-1} + b_1 z^{N-2} + \cdots + b_M z^{N-M-1}}{z^N + a_1 z^{N-1} + \cdots + a_N}$$

is proper.

Inverse z-Transform by Partial-Fraction Expansion

(1) Distinct poles. Suppose that the poles $p_1, p_2, \ldots, p_N$ are all different.

$$\frac{X(z)}{z} = \frac{A_1}{z - p_1} + \frac{A_2}{z - p_2} + \cdots + \frac{A_N}{z - p_N}$$

We want to determine the coefficients $A_1, A_2, \ldots, A_N$.

$$\frac{(z - p_k)X(z)}{z} = \frac{(z - p_k)A_1}{z - p_1} + \cdots + A_k + \cdots + \frac{(z - p_k)A_N}{z - p_N}$$

Therefore,

$$A_k = \left. \frac{(z - p_k)X(z)}{z} \right|_{z = p_k}, \qquad k = 1, 2, \ldots, N$$

(In addition, if $p_2 = p_1^*$, $A_2 = A_1^*$.)

Inverse z-Transform by Partial-Fraction Expansion

(2) Multiple-order poles. $X(z)$ has a pole of multiplicity m, that is, it contains in its denominator the factor $(z - p_k)^m$.

The partial-fraction expansion must contain the terms

$$\frac{A_{1k}}{(z-p_k)} + \frac{A_{2k}}{(z-p_k)^2} + \cdots + \frac{A_{mk}}{(z-p_k)^m}$$

Therefore,

$$A_{mk} = \left.\frac{(z-p_k)^m X(z)}{z}\right|_{z=p_k}$$

$$A_{(m-1)k} = \frac{d}{dz}\left[\frac{(z-p_k)^m X(z)}{z}\right]_{z=p_k}, \cdots$$

$$A_{1k} = \frac{d^{(m-1)}}{dz^{(m-1)}}\left[\frac{(z-p_k)^m X(z)}{z}\right]_{z=p_k}$$

Inverse z-Transform by Partial-Fraction Expansion

$$\frac{X(z)}{z} = \frac{A_1}{z - p_1} + \frac{A_2}{z - p_2} + \cdots + \frac{A_N}{z - p_N}$$

$$X(z) = \frac{A_1}{1 - p_1 z^{-1}} + \frac{A_2}{1 - p_2 z^{-1}} + \cdots + \frac{A_N}{1 - p_N z^{-1}}$$

$$\mathcal{Z}^{-1}\left\{\frac{1}{1 - p_k z^{-1}}\right\} = \begin{cases} (p_k)^n u(n), & \mathrm{ROC}: |z| > |p_k| \; (causal) \\ -(p_k)^n u(-n-1), & \mathrm{ROC}: |z| < |p_k| \; (anticausal) \end{cases}$$

Inverse z-Transform by Partial-Fraction Expansion

In the case of a double pole:

$$\begin{aligned}\frac{X(z)}{z} &= \frac{A}{(z-p)^2} + \cdots \\ X(z) &= \frac{Az^{-1}}{(1-pz^{-1})^2} + \cdots\end{aligned}$$

$$\mathcal{Z}^{-1}\left\{\frac{pz^{-1}}{(1-pz^{-1})^2}\right\} = \begin{cases} np^n u(n), & \mathrm{ROC}: |z| > |p| \ (\mathit{causal}) \\ -np^n u(-n-1), & \mathrm{ROC}: |z| < |p| \ (\mathit{anticausal}) \end{cases}$$

Decomposition of Rational z-Transform

$$X(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{1 + \sum_{k=1}^{N} a_k z^{-k}} = b_0 \frac{\prod_{k=1}^{M}(1 - z_k z^{-1})}{\prod_{k=1}^{N}(1 - p_k z^{-1})}$$

With real signals,

$$\begin{aligned} X(z) &= \sum_{k=0}^{M-N} \gamma_k z^{-k} + \sum_{k=1}^{K_1} \frac{\beta_k}{1 + \alpha_k z^{-1}} + \sum_{k=1}^{K_2} \frac{\beta_{0k} + \beta_{1k} z^{-1}}{1 + \alpha_{1k} z^{-1} + \alpha_{2k} z^{-2}} \\ &= v_0 \prod_{k=1}^{K_1} \frac{1 + v_k z^{-1}}{1 + u_k z^{-1}} \prod_{k=1}^{K_2} \frac{1 + v_{1k} z^{-1} + v_{2k} z^{-2}}{1 + u_{1k} z^{-1} + u_{2k} z^{-2}} \end{aligned}$$

where $K_1 + 2K_2 = N$.

Coefficients $\alpha_k, \beta_k, \gamma_k, u_k, v_k$ are real.

Zero-pole systems represented by linear constant-coefficient difference equations with arbitrary initial conditions.

$$H(z) = \frac{B(z)}{A(z)}$$

Assume that the input signal $x(n)$ has a rational z-transform $X(z)$

$$X(z) = \frac{N(z)}{Q(z)}$$

The system is initially relaxed, i.e. $y(-1) = y(-2) = \cdots y(-N) = 0$.

$$Y(z) = H(z)X(z) = \frac{B(z)N(z)}{A(z)Q(z)}$$

Analysis of LTI Systems in the z-Domain

Suppose that the system contains simple poles $p_1, p_2, \ldots, p_N$ and the z-transform of the input signal contains poles $q_1, q_2, \ldots, q_L$, where $p_k \neq q_m$ for all k and m.
In addition, suppose that there is no pole-zero cancellation.

A partial-fraction expansion of $Y(z)$ yields

$$Y(z) = \sum_{k=1}^{N} \frac{A_k}{1 - p_k z^{-1}} + \sum_{k=1}^{L} \frac{Q_k}{1 - q_k z^{-1}}$$

Inverse transform of $Y(z)$:

$$y(n) = \underbrace{\sum_{k=1}^{N} A_k (p_k)^n u(n)}_{\text{natural response}} + \underbrace{\sum_{k=1}^{L} Q_k (q_k)^n u(n)}_{\text{forced response}}$$

Transient Response and Steady-State Response

$$y_{nr}(n) = \sum_{k=1}^{N} A_k (p_k)^n u(n)$$

If $|p_k| < 1$ for all k, then $y_{nr}(n)$ decays to zero as n approaches infinity. The natural response is called the transient response.

$$y_{fr}(n) = \sum_{k=1}^{L} Q_k (q_k)^n u(n)$$

If the poles fall on the unit circle and consequently, the forced response persists for all $n > 0$. The forced response is called the steady-state response of the system.

Causality

Causal LTI system: $h(n) = 0, \; n < 0$.

(The ROC of the z-transform of a causal sequence is the exterior of a circle.)

A LTI system is causal *iff* the ROC of the system function is the exterior of a circle of radius $r < \infty$, including the point $z = \infty$.

Stability

BIBO stable LTI system: $\sum_{n=-\infty}^{\infty} |h(n)| < \infty$.

$$\begin{aligned} H(z) &= \sum_{n=-\infty}^{\infty} h(n) z^{-n} \\ |H(z)| &\leq \sum_{n=-\infty}^{\infty} |h(n) z^{-n}| \\ &= \sum_{n=-\infty}^{\infty} |h(n)||z^{-n}| \end{aligned}$$

When evaluated on the unit circle, i.e. $|z| = 1$,

$$|H(z)| \leq \sum_{n=-\infty}^{\infty} |h(n)| < \infty \ \Rightarrow \text{The ROC includes the unit circle.}$$

Causality and Stability

A causal and stable LTI system must have a system function converges for $|z| > r$, where $r < 1$.

A causal LTI system is BIBO stable *iff* all the poles of $H(z)$ are inside the unit circle.

cf. A causal LTI system with a rational transfer function $H(s)$ is stable *iff* all poles of $H(s)$ are in the left half of the s-plane, i.e., the real parts of all poles are negative.

Causality and Stability Example

A LTI system is characterized by the system function

$$
\begin{aligned}
H(z) &= \frac{3 - 4z^{-1}}{1 - 3.5z^{-1} + 1.5z^{-2}} \\
&= \frac{1}{1 - 0.5z^{-1}} + \frac{2}{1 - 3z^{-1}}
\end{aligned}
$$

Specify the ROC of $H(z)$ and determine $h(n)$ for the following conditions:

(1) The system is stable.
(2) The system is causal.
(3) The system is anticausal.

Causality and Stability Example

Solution. The system has poles at $z = 0.5$ and $z = 3$.

(1) Since the system is stable, its ROC must include the unit circle and hence it is $0.5 < |z| < 3$.

$$h(n) = (0.5)^n u(n) - 2(3)^n u(-n-1) \;\Rightarrow \text{noncausal}$$

(2) Since the system is causal, its ROC is $|z| > 3$.

$$h(n) = (0.5)^n u(n) + 2(3)^n u(n) \;\Rightarrow \text{unstable}$$

(3) Since the system is anticausal, its ROC is $|z| < 0.5$.

$$h(n) = -(0.5)^n u(-n-1) - 2(3)^n u(-n-1) \;\Rightarrow \text{unstable}$$

Pole-Zero Cancellation

Pole-zero cancellations can occur either in the system function itself or in the product of the system function $H(z)$ with the z-transform of the input signal $X(z)$.