ELC 4351: Digital Signal Processing

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Discrete-time Signals and Systems

Elementary Discrete-time Signals

1. Unit sample sequence

$$\delta(n) = \begin{cases} 1, & n = 0 \\ 0, & n \neq 0 \end{cases}$$

2. Unit step signal

$$u(n) = \begin{cases} 1, & n \geq 0 \\ 0, & n < 0 \end{cases}$$

3. Unit ramp signal

$$u_r(n) = \begin{cases} n, & n \geq 0 \\ 0, & n < 0 \end{cases}$$

4. Exponential signal

$$x(n) = a^n = (re^{j\theta})^n = r^n e^{j\theta n}$$

Energy signals vs. power signals

Energy: $E = \sum_{n=-\infty}^{\infty} |x(n)|^2$.

If E is finite, $0 < E < \infty$, $x(n)$ is energy signal.

Power: $P = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x(n)|^2 = \lim_{N\to\infty} \frac{1}{2N+1} E_N$.

E finite $\Rightarrow$ $P = 0$.

If P is finite, $0 < P < \infty$, $x(n)$ is power signal.

Classification of Discrete-time Signals

Periodic signals vs. aperiodic signals

$x(n)$ is periodic with period $N > 0$ iff

$$x(n+N) = x(n), \forall n.$$

The smallest N is the fundamental period.

e.g. $x(n) = A\sin(2\pi fn)$, $f = \frac{k}{N}$.

Power: $P = \frac{1}{N}\sum_{n=0}^{N-1} |x(n)|^2$.

Therefore, periodic signals are power signals.

Classification of Discrete-time Signals

Symmetric (even) vs. antisymmetric (odd) signals

Even: $x(-n) = x(n)$
Odd: $x(-n) = -x(n)$

Any signal can be expressed as a sum of an even signal and an odd signal.

$$x(n) = x_e(n) + x_o(n)$$

Proof.
$x_e(n) = \frac{1}{2}[x(n) + x(-n)]$ and $x_o(n) = \frac{1}{2}[x(n) - x(-n)]$.

Simple Manipulations of Discrete-time Signals

Time-delay: $TD_k[x(n)] = x(n-k),\ k > 0.$

Folding: $FD[x(n)] = x(-n).$

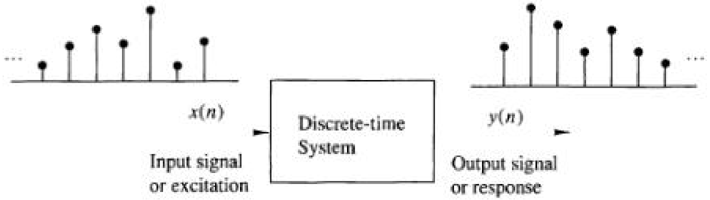
Amplitude scaling: $y(n) = Ax(n),\ -\infty < n < \infty.$

Sum: $y(n) = x_1(n) + x_2(n).$

Product: $y(n) = x_1(n)x_2(n).$ (sample-to-sample basis)

Discrete-time System

$$y(n) = \mathcal{T}[x(n)]$$



Input-Output Description of Systems

$$x(n) \xrightarrow{\mathcal{T}} y(n) \qquad\qquad y(n) = \mathcal{T}[x(n)]$$

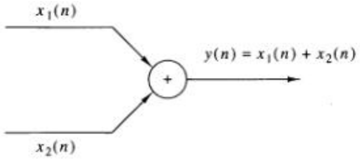
For example, an accumulator:

$$\begin{aligned} y(n) &= \sum_{k=-\infty}^{n} x(k) \\ &= x(n) + x(n-1) + x(n-2) + \cdots \\ &= \sum_{k=-\infty}^{n-1} x(k) + x(n) \\ &= y(n-1) + x(n) \end{aligned}$$

Initially relaxed at n_0: $y(n_0 - 1) = 0$.

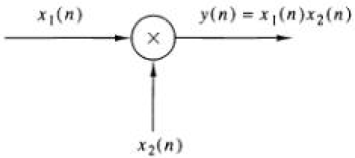
Block Diagram Representation of Discrete-time Systems

Adder



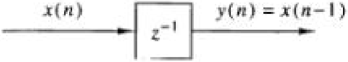
Constant Multiplier

Signal Multiplier

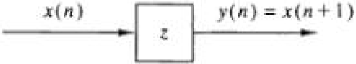


Block Diagram Representation of Discrete-time Systems

Unit Delay Element



Unit Advance Element



Classification of Discrete-time Systems

Static vs. dynamic systems

Static (memoryless):

$$\begin{aligned} y(n) &= \alpha x(n) \\ y(n) &= n^2 x(n) + \beta x^2(n) \end{aligned}$$

Dynamic:

$$\begin{aligned} y(n) &= x(n) + 3x(n-1) \\ y(n) &= \sum_{k=0}^{\infty} x(n-k) \end{aligned}$$

Classification of Discrete-time Systems

Time-invariant vs. time-variant systems

Time-invariant:

$$x(n) \xrightarrow{\mathcal{T}} y(n) \quad \text{implies} \quad x(n-k) \xrightarrow{\mathcal{T}} y(n-k).$$

$$y(n,k) = \mathcal{T}[x(n-k)] = y(n-k)$$

Linear vs. nonlinear systems

Linear system iff

$$\mathcal{T}[\alpha_1 x_1(n) + \alpha_2 x_2(n)] = \alpha_1 \mathcal{T}[x_1(n)] + \alpha_2 \mathcal{T}[x_2(n)]$$

Superposition: Scaling (multiplicative) property + Additive property

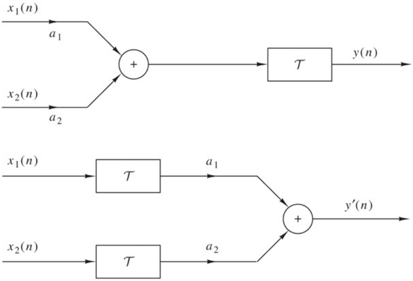


Figure 2.2.9 Graphical representation of the superposition principle. $\mathcal{T}$ is linear if and only if $y(n) = y'(n)$.

Classification of Discrete-time Systems

Causal vs. noncausal systems

Causal system iff

$$y(n) = \mathcal{T}[x(n), x(n-1), x(n-2), \cdots]$$

Classification of Discrete-time Systems

Stable vs. unstable systems

Bounded input - bounded output (BIBO) stable iff

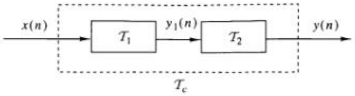
$$|x(n)| \leq M_x < \infty \Rightarrow |y(n)| \leq M_y < \infty, \forall n.$$

Interconnection of Discrete-time Systems

Cascade:

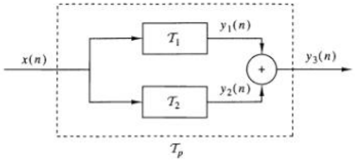
$$y(n) = \mathcal{T}_2[\mathcal{T}_1[x(n)]], \;\; \mathcal{T}_c = \mathcal{T}_2\mathcal{T}_1$$

In general, $\mathcal{T}_2\mathcal{T}_1 \neq \mathcal{T}_1\mathcal{T}_2$.



Parallel:

$$y(n) = \mathcal{T}_1[x(n)] + \mathcal{T}_2[x(n)], \;\; \mathcal{T}_p = \mathcal{T}_1 + \mathcal{T}_2$$



Techniques for Analysis of Linear Time-invariant Systems

For LTI systems, a general form of the input-output relationship.

$$y(n) = -\sum_{k=1}^{N} a_k y(n-k) + \sum_{k=0}^{M} b_k x(n-k)$$

A difference equation

Techniques for Analysis of Linear Time-invariant Systems

We use $x(n) = \sum_k c_k x_k(n)$, where $x_k(n)$ are the elementary signal components.

Suppose that $y_k(n) = \mathcal{T}[x_k(n)]$, we have

$$\begin{aligned} y(n) &= \mathcal{T}[x(n)] = \mathcal{T}\left[\sum_k c_k x_k(n)\right] \\ &= \sum_k c_k \mathcal{T}[x_k(n)] = \sum_k c_k y_k(n) \end{aligned}$$

It is chosen that, e.g.

$$x_k = e^{j\omega_k n}, \qquad k = 0, 1, \ldots, N-1.$$

where, $\omega_k = \frac{2\pi k}{N}$. $\{\omega_k\}$ are harmonically related. $\frac{2\pi}{N}$ is the fundamental frequency.

Resolution of a Discrete-time Signal into Impulses

We choose

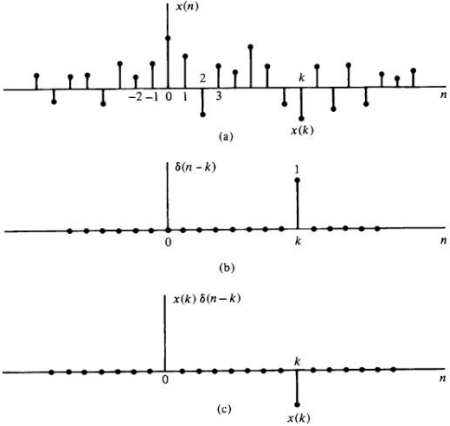
$$x_k(n) = \delta(n-k)$$

$$x(n)\delta(n-k) = x(k)\delta(n-k)$$

Therefore,

$$\begin{aligned} x(n) &= \sum_{k=-\infty}^{\infty} x(k)\delta(n-k) \\ &= \sum_{k=-\infty}^{\infty} x(k)x_k(n) \end{aligned}$$

Resolution of a Discrete-time Signal into Impulses



Response of LTI Systems to Arbitrary Inputs

$$h(n, k) \equiv \mathcal{T}[\delta(n-k)]$$

We use $x(n) = \sum_{k=-\infty}^{\infty} x(k)\delta(n-k)$.

$$\begin{aligned} y(n) &= \mathcal{T}[x(n)] = \sum_{k=-\infty}^{\infty} x(k)\mathcal{T}[\delta(n-k)] \\ &= \sum_{k=-\infty}^{\infty} x(k)h(n, k) \end{aligned}$$

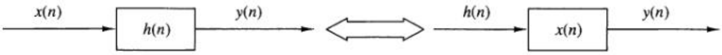
Time-invariant: $h(n) = \mathcal{T}[\delta(n)] \Rightarrow h(n, k) = h(n-k) = \mathcal{T}[\delta(n-k)]$

$$y(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k)$$

The convolution sum

The convolution sum

$$
\begin{aligned}
y(n) &= x(n) \otimes h(n) \\
&= \sum_{k=-\infty}^{\infty} x(k)h(n-k) \\
&= \sum_{k=-\infty}^{\infty} h(k)x(n-k) \\
&= h(n) \otimes x(n)
\end{aligned}
$$



Properties of Convolution and Interconnection of Systems

Identity and Shifting Properties

$$\begin{aligned} y(n) &= x(n) \otimes \delta(n) = x(n) \\ y(n-k) &= x(n) \otimes \delta(n-k) = x(n-k) \end{aligned}$$

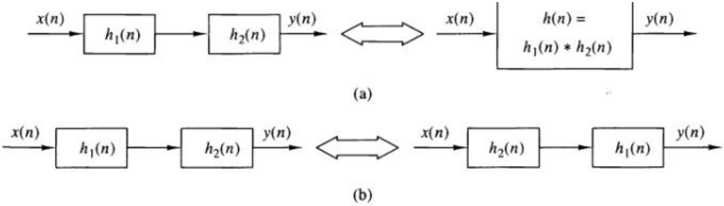
Properties of Convolution and Interconnection of Systems

Commutative Law

$$x(n) \otimes h(n) = h(n) \otimes x(n)$$

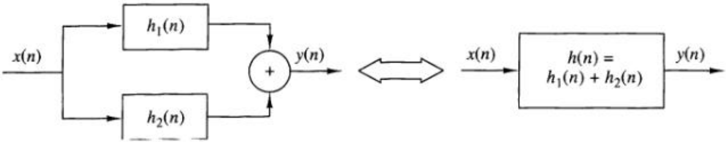
Associative Law

$$[x(n) \otimes h_1(n)] \otimes h_2(n) = x(n) \otimes [h_1(n) \otimes h_2(n)]$$



Distributive Law

$$x(n) \otimes [h_1(n) + h_2(n)] \quad = \quad x(n) \otimes h_1(n) + x(n) \otimes h_2(n)$$



Causal Linear Time-Invariant Systems

$$
\begin{aligned}
y(n_0) &= \sum_{k=-\infty}^{\infty} h(k)x(n_0-k) \\
&= \sum_{k=0}^{\infty} h(k)x(n_0-k) + \underbrace{\sum_{k=-\infty}^{-1} h(k)x(n_0-k)}_{\tilde{y}(n)}
\end{aligned}
$$

The second part $\tilde{y}(n)$ depends on the future (w.r.t. n_0) inputs $x(n_0+1), x(n_0+2), \ldots$ It has to be zero for a causal LTI system.

Therefore, the impulse response of the system must satisfy the condition

$$
\boxed{h(n) = 0, \;\; n < 0}
$$

An LTI system is causal iff its impulse response is zero for negative values of n.

Causal Linear Time-Invariant Systems

$$\boxed{h(n) = 0, \;\; n < 0}$$

$$\begin{aligned} y(n) &= \sum_{k=0}^{\infty} h(k)x(n-k) \\ &= \sum_{k=-\infty}^{n} x(k)h(n-k) \end{aligned}$$

Stability of Linear Time-Invariant Systems

If $x(n)$ is bounded, $|x(n)| \leq M_x < \infty, \forall n$.
If $y(n)$ is bounded, $|y(n)| \leq M_y < \infty, \forall n$.

$$
\begin{aligned}
y(n) &= \sum_{k=-\infty}^{\infty} h(k)x(n-k) \\
|y(n)| &= \left| \sum_{k=-\infty}^{\infty} h(k)x(n-k) \right| \\
&\leq \sum_{k=-\infty}^{\infty} |h(k)||x(n-k)| \\
&\leq M_x \sum_{k=-\infty}^{\infty} |h(k)|
\end{aligned}
$$

Stability of Linear Time-Invariant Systems

We observe that, for $|y(n)| < \infty$, a sufficient condition is

$$\boxed{\sum_{k=-\infty}^{\infty} |h(k)| < \infty}$$

It turns out this condition is not only sufficient but also necessary to ensure the stability of the system.

A LTI system is stable iff its impulse response is absolutely summable.

Systems with Finite-Duration and Infinite-Duration Impulse Response

A finite-duration impulse response (FIR) system has an impulse response that is zero outside of some finite time interval.

$$\boxed{h(n) = 0, \;\; n < 0 \;\; \text{and} \;\; n \geq M}$$

$$y(n) = \sum_{k=0}^{M-1} h(k)x(n-k)$$

An infinite-duration impulse response (IIR) system has an infinite-duration impulse response.

$$y(n) = \sum_{k=0}^{\infty} h(k)x(n-k)$$

where causality is assumed.

Implementation of Discrete-time Systems

For example, a first-order system described by the linear constant-coefficient difference equation.

$$y(n) = -a_1 y(n-1) + b_0 x(n) + b_1 x(n-1)$$

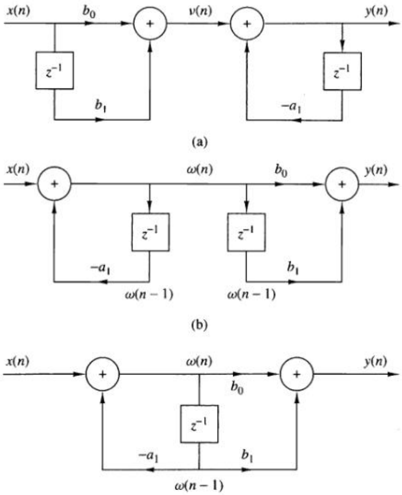
(1) Use a nonrecursive system followed by a recursive system:

$$\begin{aligned} v(n) &= b_0 x(n) + b_1 x(n-1) \\ y(n) &= -a_1 y(n-1) + v(n) \end{aligned}$$

(2) Use a recursive system followed by a nonrecursive system:

$$\begin{aligned} w(n) &= -a_1 w(n-1) + x(n) \\ y(n) &= b_0 w(n) + b_1 w(n-1) \end{aligned}$$

Implementation of Discrete-time Systems

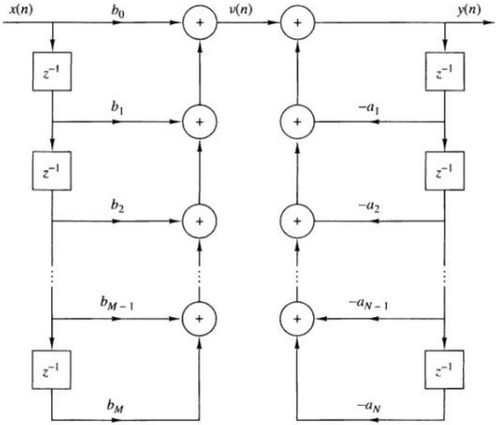


$$y(n) = -\sum_{k=1}^{N} a_k y(n-k) + \sum_{k=0}^{M} b_k x(n-k)$$

(1) Direct form I structure:

$$\begin{aligned} v(n) &= \sum_{k=0}^{M} b_k x(n-k) \\ y(n) &= -\sum_{k=1}^{N} a_k y(n-k) + v(n) \end{aligned}$$

Direct Form I Structure

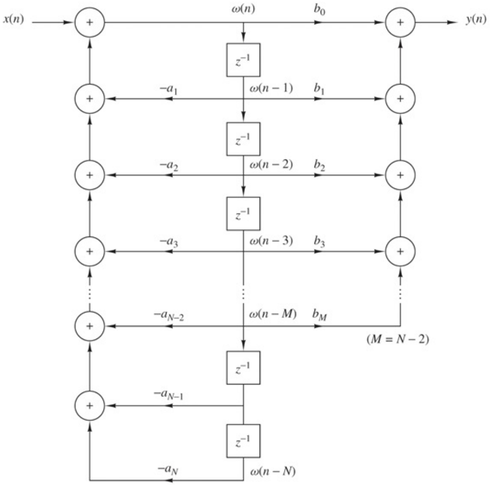


$$y(n) = -\sum_{k=1}^{N} a_k y(n-k) + \sum_{k=0}^{M} b_k x(n-k)$$

(2) Direct form II structure:

$$\begin{aligned} w(n) &= -\sum_{k=1}^{N} a_k w(n-k) + x(n) \\ y(n) &= \sum_{k=0}^{M} b_k w(n-k) \end{aligned}$$

Direct Form II Structure



Correlation of Discrete-time Signals

Crosscorrelation of sequences $x(n)$ and $y(n)$ is a sequence $r_{xy}(l)$ defined as

$$\begin{aligned} r_{xy}(l) &= \sum_{n=-\infty}^{\infty} x(n)y(n-l), \ l = 0, \pm 1, \pm 2, \ldots \\ &= \sum_{n=-\infty}^{\infty} x(n+l)y(n), \ l = 0, \pm 1, \pm 2, \ldots \end{aligned}$$

where index l is the time shift or lag.

$$r_{xy}(l) = r_{yx}(-l)$$

$$r_{xy}(l) = x(l) \otimes y(-l)$$

Correlation of Discrete-time Signals

Autocorrelation of sequence $x(n)$ is a sequence $r_{xx}(l)$ defined as

$$
\begin{aligned}
r_{xx}(l) &= \sum_{n=-\infty}^{\infty} x(n)x(n-l), \;\; l = 0, \pm 1, \pm 2, \ldots \\
&= \sum_{n=-\infty}^{\infty} x(n+l)x(n), \;\; l = 0, \pm 1, \pm 2, \ldots
\end{aligned}
$$

where index l is the time shift or lag.

$$r_{xx}(l) = r_{xx}(-l)$$

$$r_{xx}(l) = x(l) \otimes x(-l)$$

Properties of Autocorrelation and Crosscorrelation Sequences

$$\begin{aligned} |r_{xx}(l)| &\leq r_{xx}(0) = E_x \\ |r_{xy}(l)| &\leq \sqrt{r_{xx}(0) r_{yy}(0)} = \sqrt{E_x E_y} \end{aligned}$$

Normalized autocorrelation sequence:

$$\rho_{xx}(l) = \frac{r_{xx}(l)}{r_{xx}(0)}, \quad |\rho_{xx}(l)| \leq 1$$

Normalized crosscorrelation sequence:

$$\rho_{xy}(l) = \frac{r_{xy}(l)}{\sqrt{r_{xx}(0) r_{yy}(0)}}, \quad |\rho_{xy}(l)| \leq 1$$

Correlation of Periodic Sequences

Crosscorrelation:

$$r_{xy}(l) = \frac{1}{N} \sum_{n=0}^{N-1} x(n)y(n-l)$$

Autocorrelation:

$$r_{xx}(l) = \frac{1}{N} \sum_{n=0}^{N-1} x(n)x(n-l)$$

Correlation of Periodic Sequences

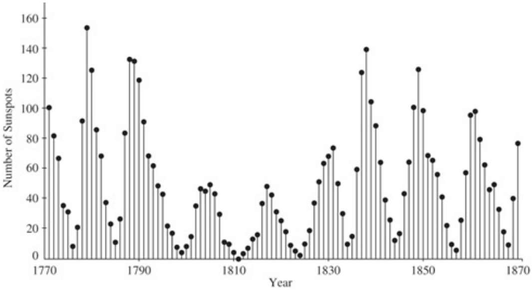
Example: Correlation is used to identify periodicity in an observed physical signal that is corrupted by random noise/interference.

$$y(n) = x(n) + w(n)$$

We observe M samples of $y(n)$, where $M \gg N$.

$$\begin{aligned}
r_{yy}(l) &= \frac{1}{M}\sum_{n=0}^{M-1} y(n)y(n-l) \\
&= \frac{1}{M}\sum_{n=0}^{M-1} [x(n)+w(n)][x(n-l)+w(n-l)] \\
&= r_{xx}(l) + r_{xw}(l) + r_{wx}(l) + r_{ww}(l)
\end{aligned}$$

Example: Identify a hidden periodicity in the Wölfer sunspot numbers in the 100-year period 1770-1869.



Input-Output Correlation Sequences

Crosscorrelation between the output and the input signal is

$$\begin{aligned} r_{yx}(l) &= y(l) \otimes x(-l) = h(l) \otimes [x(l) \otimes x(-l)] \\ &= h(l) \otimes r_{xx}(l) \end{aligned}$$

Autocorrelation of the output signal is

$$\begin{aligned} r_{yy}(l) &= y(l) \otimes y(-l) \\ &= [h(l) \otimes x(l)] \otimes [h(-l) \otimes x(-l)] \\ &= [h(l) \otimes h(-l)] \otimes [x(l) \otimes x(-l)] \\ &= r_{hh}(l) \otimes r_{xx}(l) \end{aligned}$$

The autocorrelation $r_{hh}(l)$ of the impulse response $h(n)$ exists if the system is stable.