ELC 4351: Digital Signal Processing

Short-Time Fourier Transform

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Short-Time Fourier Transform

Short-time Fourier transmforms (STFTs) divide a longer time signal into shorter segments of equal length and compute the Fourier transform separately on each shorter segment.

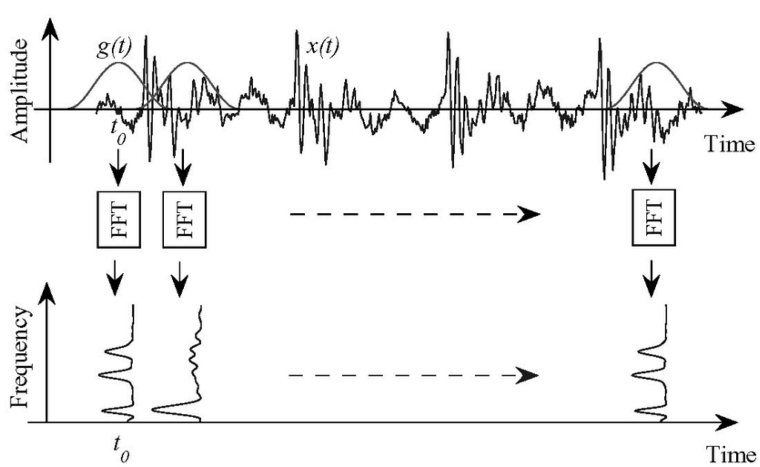


Figure: STFT on test signal $x(t)$. Here, $g(t)$ is the window function.

Continuous-Time STFT

With a sliding window function $w(t) = g(t)$, the STFT of signal $x(t)$ is

$$\boxed{X(\tau, \omega) = \int_{\infty}^{\infty} x(t)g(t-\tau)e^{-j\omega t}dt}$$

- $g(\tau)$ is the window function, e.g., a Hann window or Gaussian window centered at zero.
- $X(\tau, \omega)$ is the Fourier transform of $x(t)g(t-\tau)$.
- $X(\tau, \omega)$ is two dimensional, with τ the time axis and ω the frequency axis.
- τ is called the "slow" time with respect to the high-resolution time t.

Discrete-Time STFT

The discrete-time STFT of signal $x[n]$ is

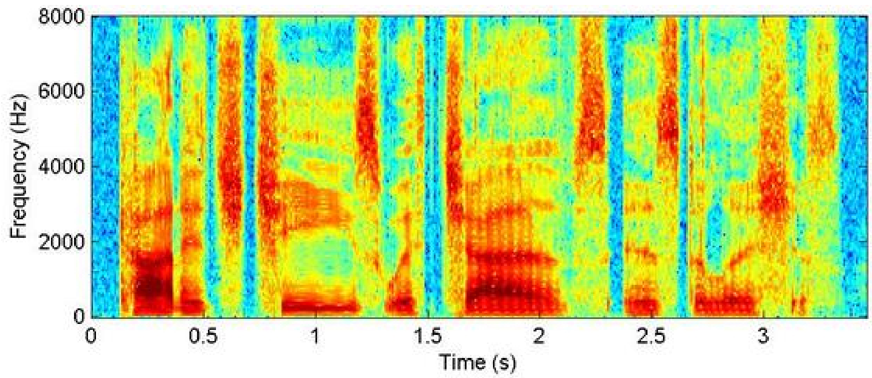
$$\boxed{X(m, \omega) = \sum_{n=-\infty}^{\infty} x[n]g[n-m]e^{-j\omega n}}$$

- m is discrete and ω is continuous.
- In practice, we use discrete-STFT. Both time and frequency variables are discrete and quantized.

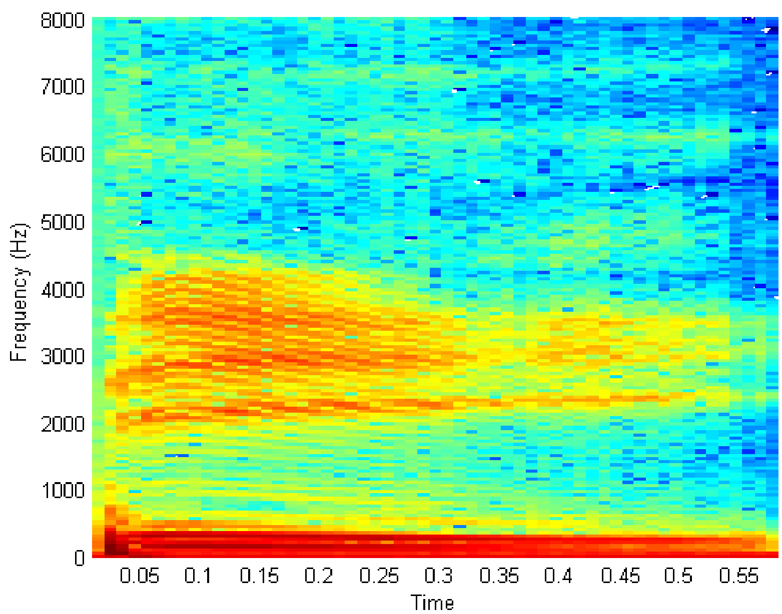
Spectrogram

The spectrogram of signal $x(t)$ is the squared magnitude of the STFT

$$\boxed{\text{Spectrogram}\{x(t)\}(\tau,\omega) = |X(\tau,\omega)|^2}$$



Spectrogram



Fourier Transform and Short-Time Fourier Transform

- The window function $g(t)$ is scaled so that

$$\int_{-\infty}^{\infty} g(\tau)d\tau = 1 \quad \Rightarrow \quad \int_{-\infty}^{\infty} g(t-\tau)d\tau = 1, \forall t$$

- Therefore,

$$x(t) = x(t)\int_{-\infty}^{\infty} g(t-\tau)d\tau = \int_{-\infty}^{\infty} x(t)g(t-\tau)d\tau$$

Fourier Transform and Short-Time Fourier Transform

- Fourier transform of $x(t)$ is

$$\begin{aligned} X(\omega) &= \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt \\ &= \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} x(t)g(t-\tau)d\tau\right]e^{-j\omega t}dt \\ &= \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} x(t)g(t-\tau)e^{-j\omega t}dt\right]d\tau \\ &= \int_{-\infty}^{\infty} X(\tau,\omega)d\tau \end{aligned}$$

- The FT is a phase-coherent sum of all of the STFTs of $x(t)$.

Continuous-Time Inverse STFT

- The inverse Fourier transform is

$$\begin{aligned} x(t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}d\omega \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} X(\tau,\omega)d\tau\right]e^{j\omega t}d\omega \\ &= \int_{-\infty}^{\infty}\left[\frac{1}{2\pi}\int_{-\infty}^{\infty} X(\tau,\omega)e^{j\omega t}d\omega\right]d\tau \end{aligned}$$

- Because $x(t) = \int_{-\infty}^{\infty} x(t)g(t-\tau)d\tau$, we have

$$x(t)g(t-\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\tau,\omega)e^{j\omega t}d\omega$$

Wavelet Transform

- The wavelet transform (WT) is another mapping from $L^2(R) \to L^2(R^2)$, but one with superior time-frequency localization as compared with the STFT.
- An admissibility condition on the wavelet is needed to ensure the invertibility of the continuous wavelet transform.
- The discrete wavelet transform (DWT) is generated by sampling the wavelet parameters (a, b) on a grid or lattice.
- A fine grid mesh permits easy reconstruction, but with oversampling. A coarse grid could result in loss of information.

Wavelets

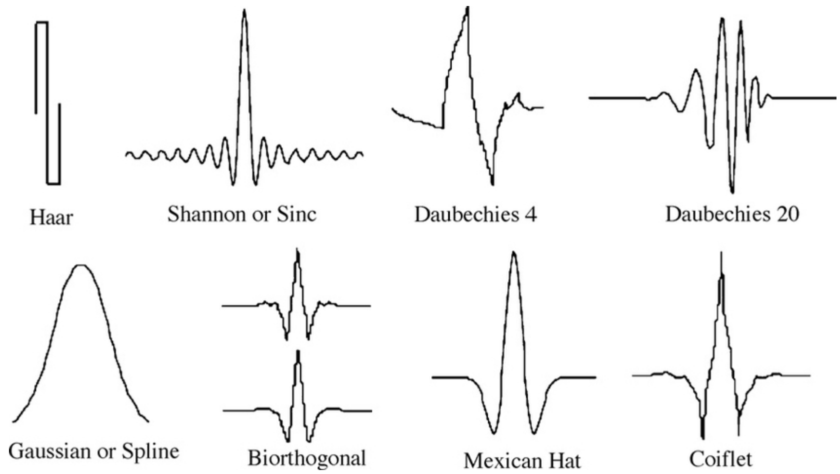


Figure: Examples of wavelet $\psi(t)$

Continuous Wavelet Transform

- The wavelet with dilation and translation of a mother wavelet function $\psi(t)$

$$\psi_{ab}(t) = \frac{1}{\sqrt{a}}\psi\left(\frac{t-b}{a}\right)$$

- The CWT maps a function $x(t)$ onto the time-scale space

$$W\{x(t)\}(a,b) = \langle x(t), \psi_{ab}(t)\rangle = \int_{-\infty}^{\infty} x(t)\psi_{ab}(t)dt$$

Continuous Wavelet Transform