ELC 4351: Digital Signal Processing

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Freq Analysis of Signals II

Frequency Analysis of Signals

Frequency-Domain and Time-Domain Signal Properties

Frequency Analysis Tools

The Fourier series	for continuous-time periodic signals
The Fourier transform	for continuous-time aperiodic signals
The Fourier series	for discrete-time periodic signals
The Fourier transform	for discrete-time aperiodic signals

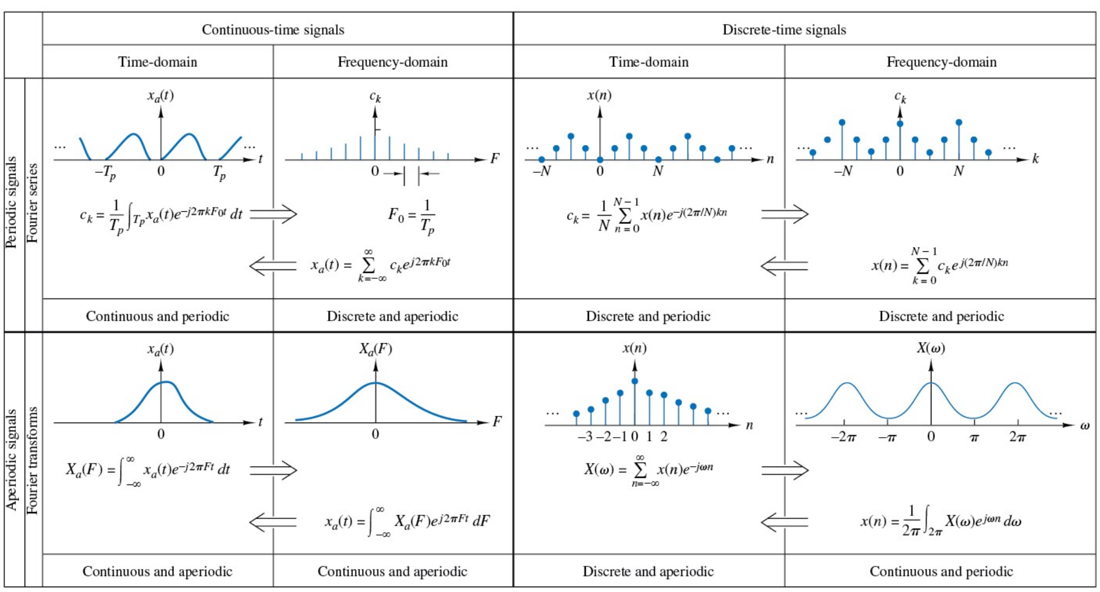
Continuous-time signals have aperiodic spectra

Discrete-time signals have periodic spectra

Periodic signals have discrete spectra

Aperiodic finite energy signals have continuous spectra

The Fourier Series for Continuous-Time Periodic Signals



Periodicity with period α in one domain implies discretization with spacing $1/\alpha$ in the other domain, and *vice versa*.

Notation

$$X(\omega) \triangleq F\{x(n)\} = \sum_{n=-\infty}^{\infty} x(n)e^{-j\omega n}$$
$$x(n) \triangleq F^{-1}\{X(\omega)\} = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(\omega)e^{j\omega n}d\omega$$

Fourier transform pair: $x(n) \overset{F}{\longleftrightarrow} X(\omega)$

where, $X(\omega)$ is periodic with period 2π.

If signal is complex, it can be expressed in rectangular form

$$x(n) = x_R(n) + jx_I(n)$$
$$X(\omega) = X_R(\omega) + jX_I(\omega)$$

Symmetry Properties of the Fourier Transform

When a signal satisfies some symmetry properties in the time domain, these properties impose some symmetry conditions on its Fourier transform.

Using the rectangular form and $e^{j\omega} = \cos\omega + j\sin\omega$, we have

$$X_R(\omega) = \sum_{n=-\infty}^{\infty} [x_R(n)\cos\omega n + x_I(n)\sin\omega n]$$
$$X_I(\omega) = -\sum_{n=-\infty}^{\infty} [x_R(n)\sin\omega n - x_I(n)\cos\omega n]$$

and

$$x_R(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} [X_R(\omega)\cos\omega n - X_I(\omega)\sin\omega n]d\omega$$
$$x_I(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} [X_R(\omega)\sin\omega n + X_I(\omega)\cos\omega n]d\omega$$

Symmetry Properties of the Fourier Transform

Real signals. $x_R(n) = x(n)$ and $x_I(n) = 0$.

$$\begin{aligned} X_R(\omega) &= \sum_{n=-\infty}^{\infty} x(n) \cos \omega n \\ X_I(\omega) &= -\sum_{n=-\infty}^{\infty} x(n) \sin \omega n \end{aligned}$$

It follows that

$$\begin{aligned} X_R(-\omega) &= X_R(\omega) \\ X_I(-\omega) &= -X_I(\omega) \end{aligned}$$

$\Longrightarrow X^*(\omega) = X(-\omega)$. The spectrum of a real signal has *Hermitian symmetry*.

Symmetry Properties of the Fourier Transform

Real signals. $x_R(n) = x(n)$ and $x_I(n) = 0$.

$$\begin{aligned} X_R(-\omega) &= X_R(\omega) \quad \text{(even)} \\ X_I(-\omega) &= -X_I(\omega) \quad \text{(odd)} \end{aligned}$$

$$\begin{aligned} |X(-\omega)| &= |X(\omega)| \quad \text{(even)} \\ \angle X(-\omega) &= -\angle X(\omega) \quad \text{(odd)} \end{aligned}$$

Symmetry Properties of the Fourier Transform

Real and even signals. $x_R(n) = x(n)$, $x_I(n) = 0$ and $x(-n) = x(n)$.

$$\begin{aligned} X_R(\omega) &= x(0) + 2\sum_{n=1}^{\infty} x(n)\cos\omega n \quad \text{(even)} \\ X_I(\omega) &= 0 \end{aligned}$$

It has real-valued spectrum, which is even function of the frequency ω.

Symmetry Properties of the Fourier Transform

Real and odd signals. $x_R(n) = x(n)$, $x_I(n) = 0$ and $x(-n) = -x(n)$.

$$\begin{aligned} X_R(\omega) &= 0 \\ X_I(\omega) &= -2\sum_{n=1}^{\infty} x(n)\sin\omega n \quad \text{(odd)} \end{aligned}$$

It has imaginary-valued spectrum, which is odd function of the frequency ω.

Symmetry Properties of the Fourier Transform

Purely imaginary signals. $x_R(n) = 0$ and $jx_I(n) = x(n)$.

$$\begin{aligned} X_R(\omega) &= \sum_{n=-\infty}^{\infty} x_I(n)\sin\omega n \quad \text{(odd)} \\ X_I(\omega) &= \sum_{n=-\infty}^{\infty} x_I(n)\cos\omega n \quad \text{(even)} \end{aligned}$$

Symmetry Properties of the Fourier Transform

Purely imaginary and odd signals. $x_R(n) = 0$, $jx_I(n) = x(n)$ and $x_I(-n) = -x_I(n)$.

$$\begin{aligned} X_R(\omega) &= 2\sum_{n=1}^{\infty} x_I(n)\sin\omega n \quad \text{(odd)} \\ X_I(\omega) &= 0 \end{aligned}$$

It has real-valued spectrum, which is odd function of the frequency ω.

Symmetry Properties of the Fourier Transform

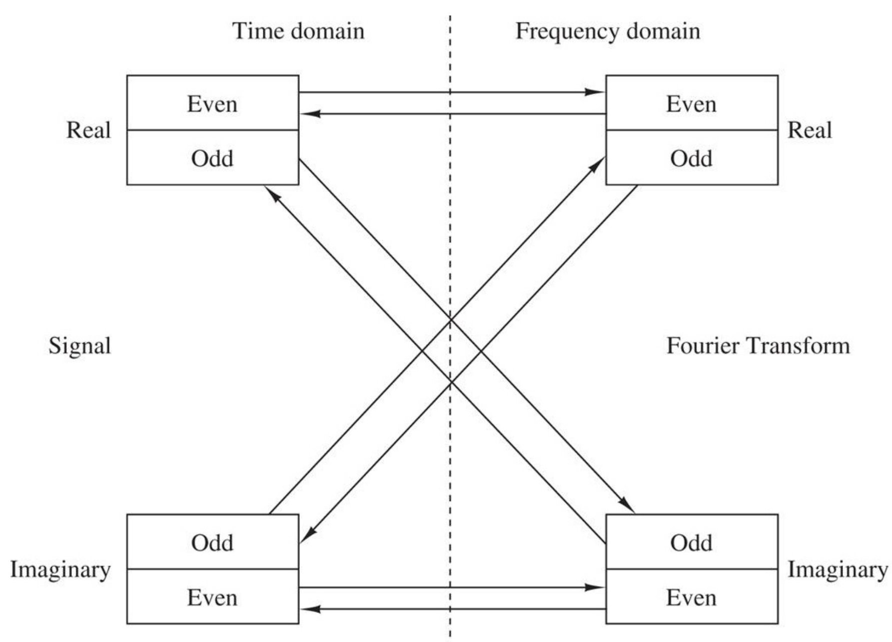
Purely imaginary and even signals. $x_R(n) = 0$, $jx_I(n) = x(n)$ and $x_I(-n) = x_I(n)$.

$$
\begin{aligned}
X_R(\omega) &= 0 \\
X_I(\omega) &= x_I(0) + 2\sum_{n=1}^{\infty} x_I(n)\cos\omega n \quad \text{(even)}
\end{aligned}
$$

It has imaginary-valued spectrum, which is even function of the frequency ω.

Symmetry Properties of the Fourier Transform

Summary of symmetry properties for the Fourier Transform



Fourier Transform Theorems and Properties

Linearity.

If $x_1(n) \longleftrightarrow X_1(\omega)$ and $x_2(n) \longleftrightarrow X_2(\omega)$,

then

$\alpha_1 x_1(n) + \alpha_2 x_2(n) \longleftrightarrow \alpha_1 X_1(\omega) + \alpha_2 X_2(\omega)$.

Fourier Transform Theorems and Properties

Time shifting.

If $x(n) \longleftrightarrow X(\omega)$,

then

$x(n-k) \longleftrightarrow e^{-j\omega k} X(\omega)$.

Fourier Transform Theorems and Properties

Time reversal.

If $x(n) \longleftrightarrow X(\omega)$,

then

$x(-n) \longleftrightarrow X(-\omega)$.

Fourier Transform Theorems and Properties

Convolution theorem.

If $x_1(n) \longleftrightarrow X_1(\omega)$ and $x_2(n) \longleftrightarrow X_2(\omega)$,

then

$x(n) = x_1(n) \otimes x_2(n) \longleftrightarrow X(\omega) = X_1(\omega)X_2(\omega)$.

Fourier Transform Theorems and Properties

Correlation theorem.

If $x_1(n) \longleftrightarrow X_1(\omega)$ and $x_2(n) \longleftrightarrow X_2(\omega)$,

then

$r_{x_1x_2}(l) \longleftrightarrow S_{x_1x_2}(\omega) = X_1(\omega)X_2(-\omega)$.

Fourier Transform Theorems and Properties

The Wiener-Khintchine theorem.

If $x(n)$ is a real signal, then $r_{xx}(l) \longleftrightarrow S_{xx}(\omega)$.

Notice that neither the autocorrelation nor the energy spectral density has any phase information.

Fourier Transform Theorems and Properties

Frequency shifting.

If $x(n) \longleftrightarrow X(\omega)$,

then

$e^{j\omega_0 n}x(n) \longleftrightarrow X(\omega - \omega_0)$.

Fourier Transform Theorems and Properties

The modulation theorem.

If $x(n) \longleftrightarrow X(\omega)$,

then

$x(n)\cos\omega_0 n \longleftrightarrow \frac{1}{2}[X(\omega + \omega_0) + X(\omega - \omega_0)]$.

Fourier Transform Theorems and Properties

Parseval's theorem.

If $x_1(n) \longleftrightarrow X_1(\omega)$ and $x_2(n) \longleftrightarrow X_2(\omega)$,

then

$$\sum_{n=-\infty}^{\infty} x_1(n)x_2^*(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X_1(\omega)X_2^*(\omega)d\omega.$$

Fourier Transform Theorems and Properties

Windowing theorem.

If $x_1(n) \longleftrightarrow X_1(\omega)$ and $x_2(n) \longleftrightarrow X_2(\omega)$,

then

$$x_1(n)x_2(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X_1(\lambda)X_2(\omega-\lambda)d\lambda.$$

Differentiation in the frequency domain.

If $x(n) \longleftrightarrow X(\omega)$,

then

$$nx(n) \longleftrightarrow j\frac{dX(\omega)}{d\omega}.$$