ELC 4351: Digital Signal Processing

Filter Design

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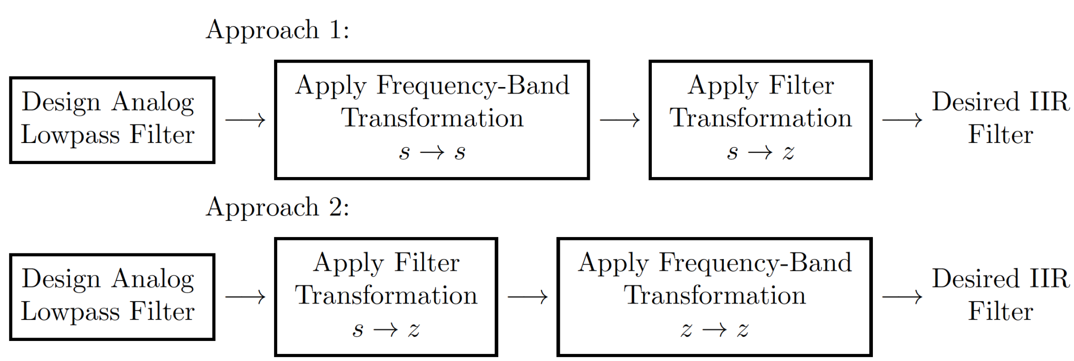
Filter Design Approach



- Here, IIR filter design is treated as magnitude-only design.
- For filter design that considers both the magnitude and phase responses, advanced optimization tools are required.

Magnitude-Squared Response of LPF

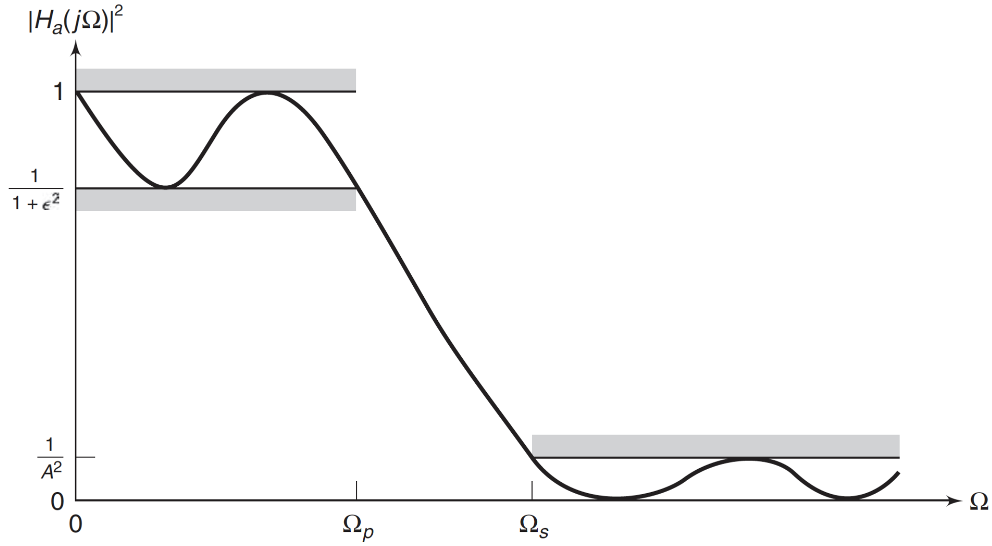


Figure: Analog lowpass filter.

LPF Specifications

- The LPF specifications on the magnitude-squared response is

$$\begin{aligned} \frac{1}{1+\epsilon^2} &\leq |H_a(j\Omega)|^2 \leq 1, \quad |\Omega| \leq \Omega_p \\ 0 &\leq |H_a(j\Omega)|^2 \leq \frac{1}{A^2}, \quad \Omega_s \leq |\Omega| \end{aligned}$$

  where ϵ is the passband ripple parameter, Ω_p is the passband cutoff frequency in rad/sec, A is the stopband attenuation parameter, and Ω_s is the stopband cutoff frequency in rad/sec.

- Therefore, $|H_a(j\Omega)|^2$ satisfies

$$\begin{aligned} |H_a(j\Omega)|^2 &= 1, \quad |\Omega| = \Omega_p \\ |H_a(j\Omega)|^2 &= \frac{1}{A^2}, \quad |\Omega| = \Omega_s \end{aligned}$$

LPF Specifications

- The parameters ϵ and A are related to parameters R_p and A_s of the dB scale as

$$R_p = -10\log_{10}\frac{1}{1+\epsilon^2} \Longrightarrow \epsilon = \sqrt{10^{R_p/10}-1}$$

$$A_s = -10\log_{10}\frac{1}{A^2} \Longrightarrow A = 10^{A_s/20}$$

- The ripples, δ_1 and δ_2, of the absolute scale are related to ϵ and A by

$$\frac{1-\delta_1}{1+\delta_1} = \sqrt{\frac{1}{1+\epsilon^2}} \Longrightarrow \epsilon = \frac{2\sqrt{\delta_1}}{1-\delta_1}$$

$$\frac{\delta_2}{1+\delta_1} = \frac{1}{A} \Longrightarrow A = \frac{1+\delta_1}{\delta_2}$$

Properties of Magnitude-Squared Response

- For the s-domain system function $H_a(s)$, we have

$$\begin{aligned} |H_a(j\Omega)|^2 &= H_a(j\Omega)H_a^*(j\Omega) \\ &= H_a(j\Omega)H_a(-j\Omega) \\ &= H_a(s)H_a(-s)|_{s=j\Omega} \end{aligned}$$

- The poles and zeros of the magnitude-squared response are distributed in a mirror-image symmetry with respect to the $j\Omega$ axis.
- For real filters, poles and zeros occur in complex conjugate pairs.

Properties of Magnitude-Squared Response

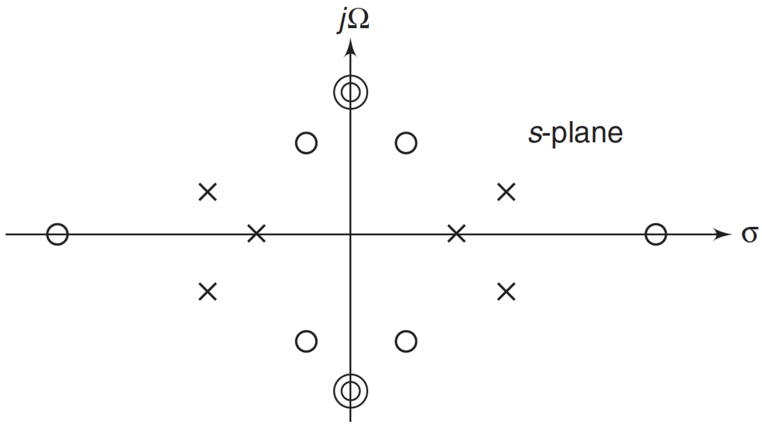


Figure: The pole-zero pattern of $H_a(s)H_a(-s)$.

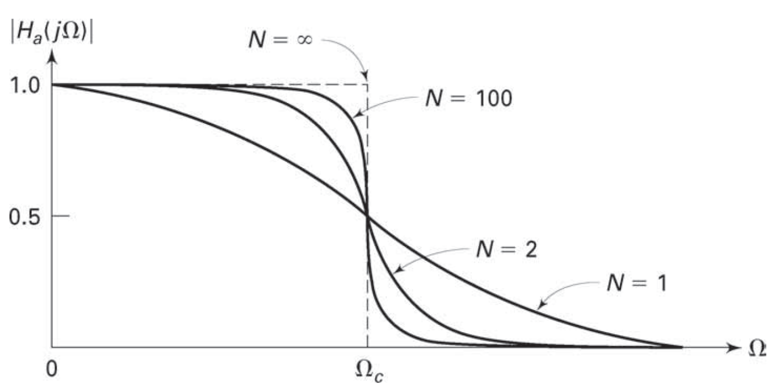
- Choose left-half poles for $H_a(s) \Longrightarrow$ Causal and Stable filter.
- Choose left-half or on-$j\Omega$-axis zeros for $H_a(s) \Longrightarrow$ Minimum-phase filter.

Prototype Analog Filters

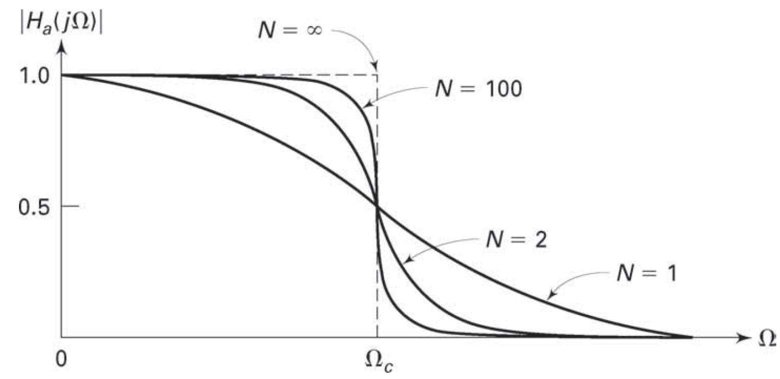
- Butterworth Lowpass Filter
  Its magnitude response is flat in both passband and stopband. The magnitude-squared response of the Nth-order lowpass filter is

$$|H_a(j\Omega)|^2 = \frac{1}{1 + \left(\frac{\Omega}{\Omega_c}\right)^{2N}}$$

where Ω_c is the cutoff frequency in rad/sec.



Butterworth Lowpass Filter



- At $\Omega = 0$, $|H_a(j0)|^2 = 1$ for all N.
- At $\Omega = \Omega_c$, $|H_a(j\Omega_c)|^2 = 0.5$ for all N. 3dB attenuation at Ω_c.
- $|H_a(j\Omega)|^2$ is a monotonically decreasing function of Ω.
- $|H_a(j\Omega)|^2$ approaches an ideal LPF as $N \to \infty$.

Butterworth Lowpass Filter

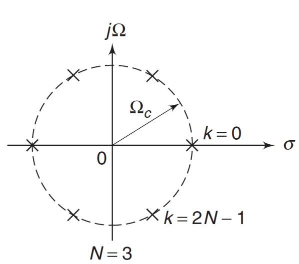
- The squared system function is

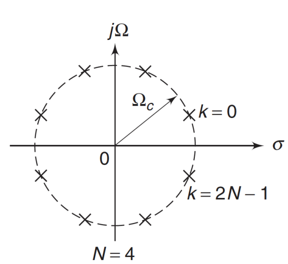
$$H_a(s)H_a(-s) = |H_a(j\Omega)|^2\Big|_{\Omega=\frac{s}{j}} = \frac{1}{1 + \left(\frac{s}{j\Omega_c}\right)^{2N}}$$

$$= \frac{(j\Omega)^{2N}}{s^{2N} + (j\Omega_c)^{2N}}$$

- The poles are

$$p_k = (-1)^{\frac{1}{2N}}(j\Omega) = \Omega_c e^{j\frac{\pi}{2N}(2k+N+1)}$$





Butterworth Lowpass Filter

- A stable and causal filter $H_a(s)$ can be specified by selecting poles in the left half-plane

$$H_a(s) = \frac{\Omega_c^N}{\prod_{\text{LHP poles}}(s - p_k)}$$

Butterworth Lowpass Filter

- At $\Omega = \Omega_p$, $-10\log_{10}|H_a(j\Omega)|^2 = R_p$

$$-10\log_{10}\left(\frac{1}{1+\left(\frac{\Omega_p}{\Omega_c}\right)^{2N}}\right) = R_p$$

- At $\Omega = \Omega_s$, $-10\log_{10}|H_a(j\Omega)|^2 = A_s$

$$-10\log_{10}\left(\frac{1}{1+\left(\frac{\Omega_s}{\Omega_c}\right)^{2N}}\right) = A_s$$

- Solving these two equations for N and Ω_c, we have

Butterworth Lowpass Filter

- We have

$$N = \left\lceil \frac{\log_{10}[(10^{R_p/10} - 1)(10^{A_s/10} - 1)]}{2\log_{10}(\Omega_p/\Omega_s)} \right\rceil$$

- To satisfy the specification exactly at Ω_p,

$$\Omega_c = \frac{\Omega_p}{\sqrt[2N]{10^{R_p/10} - 1}}$$

Or, to satisfy the specification exactly at Ω_s,

$$\Omega_c = \frac{\Omega_s}{\sqrt[2N]{10^{A_s/10} - 1}}$$

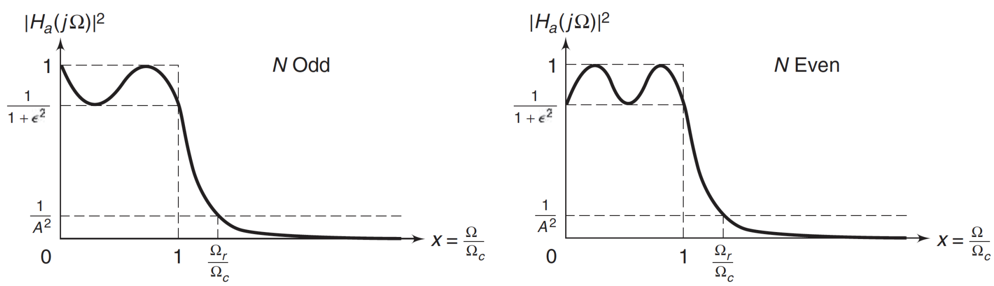
Chebyshev Lowpass Filter

- The magnitude-squared response of the Chebyshev-I (equiripple response in the passband) filter is

$$|H_a(j\Omega)|^2 = \frac{1}{1 + \epsilon^2 T_N^2\left(\frac{\Omega}{\Omega_c}\right)}$$

where N is the filter order, ϵ is the passband ripple factor (related to R_p), and T_N is the Nth-order Chebyshev polynomial given by

$$T_N(x) = \begin{cases} \cos(N\cos^{-1}(x)), & 0 \le x \le 1 \\ \cosh(\cosh^{-1}(x)), & 1 < x < \infty \end{cases} \qquad x = \Omega/\Omega_c$$

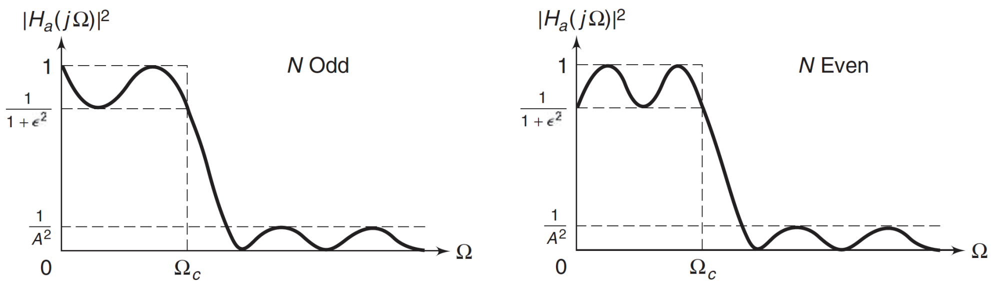


Elliptic Lowpass Filter

- Equiripple behavior in the passband and in the stopband. Achieve minimum order N for the given specifications.
- The magnitide-squared response of elliptic filter is

$$|H_a(j\Omega)|^2 = \frac{1}{1 + \epsilon^2 U_N^2\left(\frac{\Omega}{\Omega_c}\right)}$$

where N is the orer, ϵ is the passband ripple (related to R_p), and $U_N(\cdot)$ is the Nth-order Jacobian elliptic function.



Analog-to-Digital Filter Transformation

- Impulse Invariance Transformation

$$h(n) = h_a(nT)$$

Parameter T is chosen so that the shape of $h_a(t)$ is "preserved" by the samples.

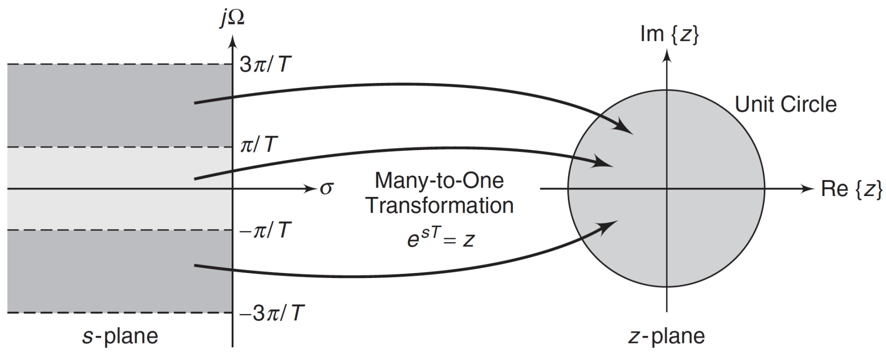
- The analog and digital frequencies are related by

$$\omega = \Omega T \quad \text{or} \quad e^{j\omega} = e^{j\Omega T}$$

- $z = e^{j\omega}$ on the unit circle and $s = j\Omega$ on the imaginary axis, we have (transforming from the s-plane to the z-plane)

$$z = e^{sT}$$

Impulse Invariance Transformation



- Many-to-one mapping: $z = e^{sT} = e^{(\sigma + j\Omega)T} = e^{\sigma T} e^{j\Omega T}$.
- $\sigma < 0$ maps into $|z| < 1$ (inside of the unit circle).
- The system functions are related through the frequency-domain aliasing formula

$$H(z) = \frac{1}{T} \sum_{k=-\infty}^{\infty} H_a\left(s - j\frac{2\pi}{T}k\right)$$

Impulse Invariance Transformation Example

- Many-to-one mapping $z = e^{sT}$.
- Design analog filter $H_a(s)$. Using partial fraction expansion, expand $H_a(s)$ into

$$H_a(s) = \sum_{k=1}^{N} \frac{R_k}{s - p_k}$$

- Transform analog poles $\{p_k\}$ into digital poles $\{e^{p_k T}\}$ to obtain the digital filter

$$H(z) = \sum_{k=1}^{N} \frac{R_k}{1 - e^{p_k T} z^{-1}}$$

Impulse Invariance Transformation Example

- Example: Transform

$$H_a(s) = \frac{s+1}{s^2+5s+6}$$

into a digital filter $H(z)$ in which $T = 0.1$.

- Solution: Expand $H_a(s)$ using partial fraction expansion

$$H_a(s) = \frac{2}{s+3} - \frac{1}{s+2}$$

The poles are at $p_1 = -3$ and $p_2 = -2$. With $T = 0.1$ we find digital poles $e^{p_k T} = e^{0.1 p_k}$. Therefore,

$$H(z) = \frac{2}{1-e^{-3T}z^{-1}} - \frac{1}{1-e^{-2T}z^{-1}} = \frac{1-0.8966z^{-1}}{1-1.5595z^{-1}+0.6065z^{-2}}$$

Bilinear Transformation

- Bilinear Transformation

$$s = \frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}} \Longrightarrow z = \frac{1+sT/2}{1-sT/2}$$

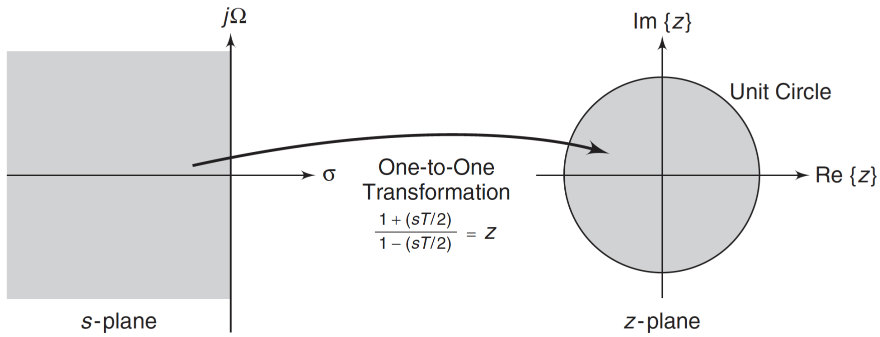
where T is a parameter.

- We have

$$\frac{T}{2}sz + \frac{T}{2}s - z + 1 = 0$$

which is bilinear in s and z.

Bilinear Transformation



- One-to-one mapping: $z = \frac{1+sT/2}{1-sT/2} = \frac{1+\sigma T/2 + j\Omega T/2}{1-\sigma T/2 - j\Omega T/2}$.
- $\sigma < 0 \implies |z| < 1$. $\sigma = 0 \implies |z| = \left|\frac{1+j\Omega T/2}{1-j\Omega T/2}\right| = 1$
- $e^{j\omega} = \frac{1+j\Omega T/2}{1-j\Omega T/2}$. Solving for ω we have

$$\omega = 2\tan^{-1}\left(\frac{\Omega T}{2}\right) \quad \text{or} \quad \Omega = \frac{2}{T}\tan\left(\frac{\omega}{2}\right)$$

Bilinear Transformation Example

- One-to-one mapping $s = \frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}}$.
- Example: Transform

$$H_a(s) = \frac{s+1}{s^2+5s+6}$$

into a digital filter $H(z)$ in which $T = 1$.

- Solution: We obtain

$$\begin{aligned} H(z) &= H_a\left(\frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}}\Big|_{T=1}\right) = H_a\left(2\frac{1-z^{-1}}{1+z^{-1}}\right) \\ &= \frac{2\frac{1-z^{-1}}{1+z^{-1}} + 1}{\left(2\frac{1-z^{-1}}{1+z^{-1}}\right)^2 + 5\left(2\frac{1-z^{-1}}{1+z^{-1}}\right) + 6} \\ &= \frac{3+2z^{-1}-z^{-2}}{20+4z^{-1}} = \frac{0.15+0.1z^{-1}-0.05z^{-2}}{1+0.2z^{-1}} \end{aligned}$$